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First Exit Times of Non-linear Dynamical Systems in $\mathbb{R}^d$ Perturbed by Multifractal Lévy Noise

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Abstract In a domain $\mathcal{G} \subset \mathbb{R}^d$ we study a dynamical system which is perturbed in finitely many directions i by one-dimensional Lévy processes with α_i-stable components. We investigate the exit behavior of the system from the domain in the small noise limit. Using probabilistic estimates on the Laplace transform of the exit time we show that it is exponentially distributed with a parameter that depends on the smallest α_i. Finally we prove that the system exits from the domain in the direction of the process with the smallest α_i.

Keywords Lévy process · Lévy flight · First exit time · Exit time law · Perturbed dynamical system · Multifractal noise

1 Introduction. Dynamical Systems Perturbed by Noise

The study of the stochastic dynamics of systems with small random perturbations has been receiving much attention in recent years. We consider a finite dimensional dynamical system generated by a vector field $b : \mathbb{R}^d \to \mathbb{R}^d$ via the differential equation $dX_t^0 = b(X_t^0)dt$. We assume that the system has at least one asymptotically stable point x. If X^0 starts in the domain of attraction of x, it tends to this point: $X_t^0 \to x$ as $t \to \infty$.

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Now perturb this deterministic system by a small random noise, i.e. consider the stochastic differential equation

$$X_t^\varepsilon = x - \int_0^t b(X_s^\varepsilon)ds + \varepsilon\psi_t, \tag{1.1}$$

where ψ is a d-dimensional random process and ε is small. As an important feature of the asymptotic behavior of the resulting stochastic dynamical system, the exit from an open set $\mathcal{G}$ contained in a domain of attraction of a stable point may be investigated. For the most extensively studied case of Gaussian perturbations the theory of large deviations is used to show that the mean exit time is exponential in the noise parameter. More precisely, the logarithmic rate of the mean exit time (Kramers' time) is proportional to $\frac{1}{\varepsilon^2}$ multiplied by the height of the point of minimal quasipotential on the boundary of $\mathcal{G}$, the latter being related to the vector field b, see [10, 17] .

A general theory of large deviations for Markov processes can be found in [23], mean exit times of Markov processes with heavy tails has been first studied in [11].

In recent time, small noise dynamics with stable Lévy noises, especially the first exit problem, has attracted interest in the context modelling of extreme events in climate [8, 12, 18], physics [4] and finance [22].

In case $d = 1$, and if Gaussian noise is replaced by Lévy noise with discontinuous trajectories, the asymptotic exit characteristics have been studied in [13–15] and in [5–7, 9, 24]. As expected, the presence of jumps allows the process a much faster exit from the domain. In the case of regularly varying tails for the jump measure, the mean exit time depends polynomially on the noise parameter.

In this paper we intend to transfer the methods developed in [14] to systems in Euclidean space with finite dimension d. We consider a domain $\mathcal{G} \subset \mathbb{R}^d$ and a dynamical system perturbed by multi-fractal Lévy processes acting in finitely many spatial directions. The result we derive retains the main feature of the one dimensional case. In the small noise limit, we show that the exit time is proportional to the time of the first jump exceeding the distance from the stable point to the boundary of the domain in the direction of the noise component corresponding to the smallest stability index.

The structure of the paper is as follows. In Sect. 2 we discuss the geometric framework for the domain $\mathcal{G}$ and the dynamical system, and fix the notation. In Sect. 3 we give a heuristic discussion of the asymptotic exit time in terms of its Laplace transform, based on a decomposition of the Lévy noise into a small and a large jump part, which is done in each of the processes' directions. Section 4 is devoted to a rigorous underpinning of the heuristic frame. We show that small jumps play a marginal role, and establish that jumps in directions of small stability indices happen much earlier than in the remaining ones. In a second part of this section we draw some corollaries concerning the places where exits occur. In the final Sect. 5 we illustrate our findings by some simple examples.

2 Preliminaries and Notation

2.1 The Dynamical System and the Domain $\mathcal{G}$

We consider a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbf{P})$. We assume that the filtration $(\mathcal{F}_t)_{t\geq 0}$ fulfills the usual conditions, i.e. it is right-continuous and consists of σ-algebras which are complete with respect to $\mathbf{P}$, see [19]. Let $\mathcal{G} \subseteq \mathbb{R}^d$ be a bounded domain with

$0 \in \mathcal{G}$. We are given the following family of stochastic differential equations:

$$X_t^\varepsilon(x) = x + \int_0^t b(X_s^\varepsilon(x))ds + \sum_{i=1}^{N+M} \varepsilon\lambda_i r_i L_t^i, \quad \varepsilon > 0,\ x \in \mathcal{G},\ t \geq 0, \tag{2.1}$$

where $N, M \in \mathbb{N}$. Later we will specify some geometrical assumptions on $\mathcal{G}$. The principal goal of our work consists in describing the small noise dynamical behavior of X^ε, i.e. its dynamical behavior as ε tends to zero. We are interested in the exit time of X^ε from the domain $\mathcal{G}$ and the directions of the exit, both as functions of the small noise parameter ε.

We compare our stochastic equation (2.1) with the corresponding deterministic equation given by

$$z_t(x) = x + \int_0^t b(z_s)ds, \quad t \geq 0. \tag{2.2}$$

Let us first discuss the noise term of (2.1). For $1 \leq i \leq N + M$ the r_i are different unit vectors, i.e. $r_i \in \mathbb{R}^d$, $\|r_i\| = 1$ and $|\langle r_i, r_j\rangle| \neq 1$ if $i \neq j$. The $\lambda_i > 0$ are just pre-factors. For $\alpha_i \in (0, 2)$, $1 \leq i \leq N + M$, satisfying

$$\alpha_1 = \alpha_2 = \cdots = \alpha_N < \alpha_{N+1} \leq \alpha_{N+2} \leq \cdots \leq \alpha_{N+M}, \tag{2.3}$$

the stochastic processes $L^i = (L_t^i)_{t\geq 0}$ are independent 1-dimensional mean-zero Lévy-processes with symmetric, α_i-stable components. Lévy processes are, in general, random processes with independent and stationary increments that are continuous in probability and have right-continuous paths with left limits. They are characterized by their infinitely divisible one-dimensional distributions, with the following Lévy-Khintchin representations

$$\mathbf{E}e^{iuL_t^j} = \exp\left(-t\frac{u^2}{2}d + t\int_{\mathbb{R}\setminus\{0\}} \left(e^{iuy} - 1 - iuy\mathbb{I}_{\{|y|<1\}}(y)\right)\frac{1}{|y|^{1+\alpha_j}}dy\right), \tag{2.4}$$

$1 \leq j \leq N + M, u \in \mathbb{R}, t \geq 0$. This means that every L^j is a sum of two independent processes, a standard Brownian motion with variance $d \geq 0$ and an α_j-stable Lévy motion with Lévy measure $\nu_j(dy) = \frac{1}{|y|^{1+\alpha_j}}dy$ for $y \neq 0$. These are heavy-tailed jump measures with infinite mass, possessing no moments of second order, since $\mathbf{E}|L_t^j|^\delta < \infty$ iff $\delta < \alpha_j$, $1 \leq j \leq N+M$. The particular arrangement of the α_j in (2.3) is chosen to allow an efficient description of the direction determining the exit dynamics from $\mathcal{G}$: the one belonging to the smallest α_j.

More information on Lévy processes is available in [2, 3, 21]. For stable processes see [16, 20].

Let us now formulate assumptions on the geometry of $\mathcal{G}$ and on the vector field b: $\overline{\mathcal{G}} \to \mathbb{R}^d$. For $1 \leq i \leq N+M$ we denote by $\Omega_i := \{x \in \mathbb{R}^d : x = t \cdot r_i \text{ for a } t \in \mathbb{R}\}$ the straight line in the direction of r_i. With a constant $C > 1$ we state the following assumptions.

(A1) For all $1 \leq i \leq N + M$ the set $\overline{\mathcal{G}} \cap \Omega_i$ is connected. Therefore we can find numbers $a_i, b_i > 0$ and a closed interval $I_i := [-a_i, b_i]$, such that for $t \in (-a_i, b_i)$ we have: $t \cdot r_i \in \mathcal{G}$. Since $0 \in \mathcal{G}$ and $\mathcal{G}$ is open, $\mathcal{G} \cap \Omega_i \neq \emptyset$.

(A2) The vector field b is smooth, i.e. $b \in C^1(\overline{\mathcal{G}})$. As a consequence b is Lipschitz continuous: for $x, y \in \overline{\mathcal{G}}$

$$\|b(x) - b(y)\| \leq C\|x - y\|. \tag{2.5}$$

(A3) The boundary $\partial\mathcal{G}$ is a C^1-manifold, so that the vector field n of the outer normals on the boundary exists. We assume

$$\langle b(z), n(z)\rangle < -\frac{1}{C}, \tag{2.6}$$

for $z \in \partial\mathcal{G}$. This means that b "points into $\mathcal{G}$".

(A4) For the linearization $b(z) = Bz + r(z)$, $z \in \overline{\mathcal{G}}$, where B is the Jacobian matrix at zero, we assume that $\lim_{\|x\|\to 0} \frac{\|r(x)\|}{\|x\|} = 0$, $r(0) = 0$, and that the real parts of the eigenvalues of B are negative and bounded above by $-\frac{1}{C}$.

(A5) Zero is an attractor of the domain, i.e. $b(0) = 0$, and for every starting value $x \in \mathcal{G}$, the deterministic solution $(z_t(x))_{t\geq 0}$ vanishes asymptotically:

$$\lim_{t\to\infty} z_t(x) = 0. \tag{2.7}$$

Since the Lévy processes L^j are semi-martingales, (2.1) is well defined and with the help of (A2) we know that there is a unique solution which possesses the strong Markov property (see [19]). For technical purposes we draw some easy conclusions from these assumptions.

We define the inner parts of $\mathcal{G}$ by $\mathcal{G}_\delta := \{z \in \mathcal{G} : \mathrm{dist}(z, \partial\mathcal{G}) \geq \delta\}$. We can find a $\delta_0 > 0$, such that if $\|x\| < \delta_0$, then $x \in \mathcal{G}$ and for all $\delta \in (0, \delta_0)$ and for the constant C from (A1)–(A4) the following is true.

(C1) From the theory of ODE and (A4) we know that 0 is an exponentially stable point, i.e. for $t \geq 0$ and $\|x\| \leq \delta_0$ we have $\|z_t(x)\| < Ce^{-\frac{1}{C}t}\|x\|$.

(C2) For $\|y\| < \delta_0$ we consider the straight lines in the direction r_i starting from y: $g_y^i(t) = y + t \cdot r_i$ for $t \in \mathbb{R}$. We denote the distance function to the boundary by:

$$d_i^+(y) := \inf\{t > 0 : g_y^i(t) \in \partial\mathcal{G}\}, \tag{2.8}$$

$$d_i^-(y) := \sup\{t < 0 : g_y^i(t) \in \partial\mathcal{G}\}. \tag{2.9}$$

With the help of (A1) and the relative compactness of $\mathcal{G}$ we infer that $g_y^i(t) \notin \mathcal{G}$ for $t \notin (d_i^-(y), d_i^+(y))$ for all $i = 1, \ldots, N + M$. Otherwise $g_y^i(t) \in \mathcal{G}$. By (A3) the distance functions are continuous. Therefore for all $\|x\|, \|y\| < \delta_0$ we have $|d_i^+(x) - d_i^+(y)| \leq C\|x - y\|$, $|d_i^-(x) - d_i^-(y)| \leq C\|x - y\|$. For abbreviation we set $d_i^+ := d_i^+(0)$ and $d_i^- = d_i^-(0)$.

(C3) For $\delta \in (0, \delta_0)$ we define δ-tubes outside of $\mathcal{G}$ by $\Omega_i^+(\delta) := \{y \in \mathbb{R}^d : \|\langle y, r_i\rangle r_i - y\| < \delta, \langle y, r_i\rangle > 0\} \cap \mathcal{G}^c$. Analogously we define $\Omega_i^-(\delta)$ by changing to $\langle y, r_i\rangle < 0$. Then we have $\Omega_i^-(\delta) \cap \Omega_j^-(\delta) \cap \mathcal{G}_\delta^c = \emptyset$ and $\Omega_i^+(\delta) \cap \Omega_j^+(\delta) \cap \mathcal{G}_\delta^c = \emptyset$ for $i \neq j$, $i, j \in \{1, \ldots, N + M\}$.

(C4) [1] states that the sets $\mathcal{G}_\delta$ are positively invariant for all $\delta \in (0, \delta_0)$, in the sense that the deterministic solutions starting in $\mathcal{G}_\delta$ do not leave this set for all times $t \geq 0$.

(C5) As a consequence of (C1), there exists a time $T_0 > 0$ with $\sup_{x\in\mathcal{G}} \inf\{t > 0 : \|z_t(x)\| \leq \delta_0\} \leq T_0$.

(C6) From (A3), we have $\sup_{x\in\mathcal{G}\backslash\mathcal{G}_\delta} \inf\{t > 0 : z_t(x) \in \mathcal{G}_\delta\} \leq C\delta$, for all $\delta \in (0, \delta_0)$, i.e. the time it takes z, started in $\mathcal{G}\backslash\mathcal{G}_\delta$, to attain $\mathcal{G}_\delta$, is of order δ.

2.2 Decomposition into Small and Large Jump Parts

The aim of this subsection is to decompose the Lévy processes L^i by means of the Lévy-Itô-decomposition into a small jump part and an independent part with large jumps. While small jumps do not contribute much, we will show that large jumps play the crucial role in the exit behavior of X^ε in the small noise limit. For $\varepsilon < 1$ and $\rho \in (0,1)$ we consider the truncated Lévy measures

$$\begin{aligned}\nu_{\xi^i}(A) &:= \nu_i\left(A \cap \left[-\frac{1}{\lambda_i \varepsilon^\rho}, \frac{1}{\lambda_i \varepsilon^\rho}\right]\right),\\ \nu_{\eta^i}(A) &:= \nu_i\left(A \cap \left[-\frac{1}{\lambda_i \varepsilon^\rho}, \frac{1}{\lambda_i \varepsilon^\rho}\right]^c\right),\end{aligned} \tag{2.10}$$

where $i \in \{1,\dots,N+M\}$ and $A \in \mathcal{B}(\mathbb{R})$. Thus, by writing the equation $L^i = \xi^i + \eta^i$ we have decomposed L^i into two processes ξ^i and η^i with generating triplets $(d, \nu_{\xi_i}, 0)$ and $(0, \nu_{\eta_i}, 0)$ respectively. The absolute values of jumps of the processes $\varepsilon\xi^i$ do not exceed the threshold $\varepsilon^{1-\rho}$.

For the finite Lévy measure ν_{η_i} and for $i = 1,\dots,N+M$ we define the jump intensity

$$\beta_i := \nu_{\eta^i}(\mathbb{R}) = \int_{\mathbb{R}\setminus[-\frac{1}{\lambda_i\varepsilon^\rho}, \frac{1}{\lambda_i\varepsilon^\rho}]} \nu_i(dx) = \frac{2}{\alpha_i}\lambda_i^{\alpha_i}\varepsilon^{\rho\alpha_i}. \tag{2.11}$$

η^i is a compound Poisson process with intensity β_i, and its jumps are distributed according to the law $\beta_i^{-1}\nu_{\eta_i}(\cdot)$. For $k \ge 0$ denote by τ_k^i the kth jumping time of η^i and by $W_k^i := \eta^i_{\tau_k^i} - \eta^i_{\tau_k^i -}$ the heights of the jumps of η^i at time τ_k^i. Then the inter-jump periods $S_k^i := \tau_k^i - \tau_{k-1}^i$ are exponentially distributed with parameter β_i. The families $((W_k^i)_{k\in\mathbb{N}}, (S_k^i)_{k\in\mathbb{N}}, (\xi_t^i)_{t\ge 0})_{1\le i\le M+N}$ are independent. For $a, b > 0$ and $\min(a,b) > \varepsilon^{1-\rho}$ we compute the probability for a jump leaving the interval $[-a,b]$ by:

$$\begin{aligned}\mathbf{P}(\varepsilon\lambda_i W_1^i \notin [-a,b]) &= \frac{1}{\beta_i}\int_{\mathbb{R}\setminus[\frac{-a}{\varepsilon\lambda_i}, \frac{b}{\varepsilon\lambda_i}]} \mathbb{I}_{\{|y|>\lambda_i^{-1}\varepsilon^{-\rho}\}}(y)\frac{dy}{|y|^{1+\alpha_i}}\\ &= \frac{\varepsilon^{\alpha_i}}{\alpha_i\beta_i}\lambda_i^{\alpha_i}\left[\frac{1}{a^{\alpha_i}} + \frac{1}{b^{\alpha_i}}\right].\end{aligned} \tag{2.12}$$

To abbreviate relevant parameters of the compound Poisson part of our dynamics we set

$$\begin{aligned}\Phi_i &:= \frac{1}{(-d_i^-)^{\alpha_i}} + \frac{1}{(d_i^+)^{\alpha_i}}, \qquad m_i := m_i(\varepsilon) := \frac{\varepsilon^{\alpha_i}}{\alpha_i}\lambda_i^{\alpha_i}\Phi_i,\\ \beta^S &:= \sum_{i=1}^{M+N}\beta_i, \qquad m_\varepsilon := \sum_{i=1}^{N} m_i = \frac{\varepsilon^{\alpha_1}}{\alpha_1}\sum_{i=1}^{N}\lambda_i^{\alpha_1}\Phi_i,\end{aligned} \tag{2.13}$$

where the $d_i^\pm$ are chosen according to (C2). Then we see that

$$\mathbf{P}(\varepsilon\lambda_i W_1^i \notin [d_i^-, d_i^+]) = \frac{m_i}{\beta_i}. \tag{2.14}$$

Now we define the time of big jumps of the process X^ε recursively for $k > 1$ starting with $\tau_0^* := 0$:

$$\tau_1^* := \bigwedge_{i=1}^{N+M}\tau_1^i, \qquad \tau_k^* := \bigwedge_{\tau_j^i > \tau_{k-1}^*}\tau_j^i. \tag{2.15}$$

Because of the independence of $\{\tau_j^i:\ i \in \{1,\dots,N+M\}, j \in \mathbb{N}\}$ we know that τ_1^* has an exponential distribution with parameter β^S. Due to the strong Markov property of X^ε the inter-jump times $S_k^* := \tau_k^* - \tau_{k-1}^*$, $k \geq 1$, have the same distribution as τ_1^*. Finally we define σ_ε the exit time of X^ε from the domain $\mathcal{G}$ by

$$\sigma_\varepsilon = \inf\{t \geq 0:\ X_t^\varepsilon \notin \mathcal{G}\}. \tag{2.16}$$

As a crucial feature of our reasoning, we have a separate view on the different directions r_i. If we compare the mean waiting times for the next jump β_i^{-1} for small ε, we see

$$\beta_i^{-1} < \beta_j^{-1} \quad \Leftrightarrow \quad \frac{\alpha_i \lambda_j^{\alpha_j}}{\alpha_j \lambda_i^{\alpha_i}} < \varepsilon^{\rho(\alpha_i - \alpha_j)} \quad \Leftrightarrow \quad \alpha_i < \alpha_j, \tag{2.17}$$

for $i \neq j$. This means that for small ε, directions i with smaller α_i jump earlier on the average. The same is true if we compare the intensity of big jumps (2.14). Again directions i with smaller α_i contribute a larger jump intensity. So we end up with the conclusion that in the limit $\varepsilon \to 0$ the process X^ε exits the domain in the direction i corresponding to the smallest α_i.

3 Heuristic Derivation of the Main Result

In this section we give a heuristic sketch of the derivation of our main results on the asymptotics of the exit time σ_ε and the place, where the exit occurs. For this purpose we calculate the Laplace transform of the normalized exit time $m_\varepsilon \sigma_\varepsilon$. We aim at showing that it belongs to a standard exponentially distributed random variable. This will justify that we can approximate the law of σ_ε by an exponential distribution with intensity m_ε.

We will show below that exits from the domain at times that do not coincide with large jumps are asymptotically negligible. If we take into account exits only at times of large jumps, we can employ the strong Markov property to argue for starting state $x \in \mathcal{G}$ and $\theta > -1$ in the following way:

$$\begin{aligned}
\mathbf{E}[e^{-\theta m_\varepsilon \sigma_\varepsilon}] &\approx \sum_{k=1}^{\infty} \mathbf{E}[e^{-\theta m_\varepsilon \sigma_\varepsilon} \mathbb{I}\{\sigma_\varepsilon = \tau_k^*\}] \\
&\approx \sum_{k=1}^{\infty} \mathbf{E}\Bigg[e^{-\theta m_\varepsilon \tau_k^*} \prod_{j=1}^{k-1} \mathbb{I}\{X_{t+\tau_{j-1}}(X_{\tau_{j-1}}) \in \mathcal{G}, t \in [0, S_j^*]\} \\
&\qquad \times \mathbb{I}\{X_{t+\tau_{k-1}}(X_{\tau_{k-1}}) \in \mathcal{G}, t \in [0, S_k^*), X_{\tau_k^*} \notin \mathcal{G}\} \Bigg] \\
&\approx \sum_{k=1}^{\infty} (\mathbf{E}[e^{-\theta m_\varepsilon \tau_1^*}(1 - \mathbb{I}\{X_{\tau_1^*} \notin \mathcal{G}\})])^{k-1} \\
&\qquad \times \mathbf{E}[e^{-\theta m_\varepsilon \tau_1^*} \mathbb{I}\{X_{\tau_1^*} \notin \mathcal{G}\}].
\end{aligned} \tag{3.1}$$

To compute the factors in the last expression we use a decomposition into the different directions:

$$\mathbf{E}[e^{-\theta m_\varepsilon \tau_1^*} \mathbb{I}\{X_{\tau_1^*} \notin \mathcal{G}\}] = \sum_{l=1}^{M+N} \mathbf{E}[e^{-\theta m_\varepsilon \tau_1^l} \mathbb{I}\{X_{\tau_1^*} \notin \mathcal{G}\} \mathbb{I}\{\tau_1^l = \tau_1^*\}]$$

$$\approx \sum_{l=1}^{M+N} \mathbf{E}[e^{-\theta m_\varepsilon \tau_1^l} \mathbb{I}\{\tau_1^l = \tau_1^*\}]$$
$$\times \mathbf{P}(\varepsilon \lambda_l W_1^l \notin [d_l^-, d_l^+])$$
$$= \sum_{l=1}^{M+N} \frac{\beta_l}{\beta^S + \theta m_\varepsilon} \frac{m_l}{\beta_l} = \frac{m_\varepsilon}{\beta^S + \theta m_\varepsilon}(1 + R_\varepsilon). \tag{3.2}$$

The term $R_\varepsilon := \frac{1}{m_\varepsilon} \sum_{i=N+1}^{M} \frac{\varepsilon^{\alpha_i}}{\alpha_i} \lambda_i^{\alpha_i} \Phi_i$ converges to zero as $\varepsilon \to 0$, since $\alpha_1 < \alpha_i$ for $i = N+1, \dots, M$. So we can conclude for the Laplace transform in the small noise limit:

$$\mathbf{E}[e^{-\theta m_\varepsilon \sigma_\varepsilon}] \approx \sum_{k=1}^{\infty} \left[\frac{\beta^S - m_\varepsilon(1 + R_\varepsilon)}{\beta^S + \theta m_\varepsilon} \right]^{k-1} \frac{m_\varepsilon}{\beta^S + \theta m_\varepsilon}(1 + R_\varepsilon)$$
$$= \frac{1 + R_\varepsilon}{\theta + 1 + R_\varepsilon} \overset{\varepsilon \downarrow 0}{\to} \frac{1}{1+\theta}. \tag{3.3}$$

With this knowledge we can derive a result on the place where the process exits from the domain. Using (C3), for $\delta \in (0, \delta_0)$ and $i \in \{1, \dots, M+N\}$ we first obtain

$$\mathbf{P}(X^\varepsilon_{\tau_1^*} \in \Omega_i^+(\delta)) = \sum_{k=1}^{N+M} \mathbf{P}(X^\varepsilon_{\tau_1^k} \in \Omega_i^+(\delta)) \mathbf{P}(\tau_1^k = \tau_1^*)$$
$$= \mathbf{P}(X^\varepsilon_{\tau_1^i} \in \Omega_i^+(\delta)) \mathbf{P}(\tau_1^i = \tau_1^*) \approx \frac{\beta_i}{\beta^S} \mathbf{P}(\varepsilon \lambda_i W_1^i \ge d_i^+)$$
$$= \frac{1}{\beta^S \alpha_i} \varepsilon^{\alpha_i} \lambda_i^{\alpha_i} (d_i^+)^{-\alpha_i}. \tag{3.4}$$

Employing the calculation that led to the Laplace transform above for $\theta = 0$ we deduce

$$\mathbf{P}(X^\varepsilon_{\sigma(\varepsilon)} \in \Omega_i^+(\delta)) \approx \sum_{k=1}^{\infty} (\mathbf{E}[e^{-\theta m_\varepsilon \tau_1^*}(1 - \mathbb{I}\{X_{\tau_1^*} \notin \mathcal{G}\})])^{k-1} \mathbf{P}(X^\varepsilon_{\tau_1^*} \in \Omega_i^+(\delta))$$
$$= \sum_{k=1}^{\infty} \left[1 - \frac{m_\varepsilon}{\beta^S}(1 + R_\varepsilon) \right]^{k-1} \frac{1}{\beta^S \alpha_i} \varepsilon^{\alpha_i} \lambda_i^{\alpha_i} (d_i^+)^{-\alpha_i}$$
$$= \frac{\frac{\varepsilon^{\alpha_i}}{\alpha_i} \lambda_i^{\alpha_i} (d_i^+)^{-\alpha_i}}{m_\varepsilon(1 + R_\varepsilon)}. \tag{3.5}$$

Since for small ε m_ε is of order ε^{α_1} we conclude

$$\mathbf{P}(X^\varepsilon_{\sigma(\varepsilon)} \in \Omega_i^+(\delta)) \approx \frac{\frac{\varepsilon^{\alpha_i}}{\alpha_i} \lambda_i^{\alpha_i} (d_i^+)^{-\alpha_i}}{m_\varepsilon(1 + R_\varepsilon)} \overset{\varepsilon \downarrow 0}{\to} \begin{cases} \frac{\lambda_i^{\alpha_1} (d_i^+)^{-\alpha_1}}{\sum_{k=1}^{N} \lambda_k^{\alpha_1} \Phi_k}, & i \in \{1, \dots, N\}, \\ 0, & \text{else.} \end{cases} \tag{3.6}$$

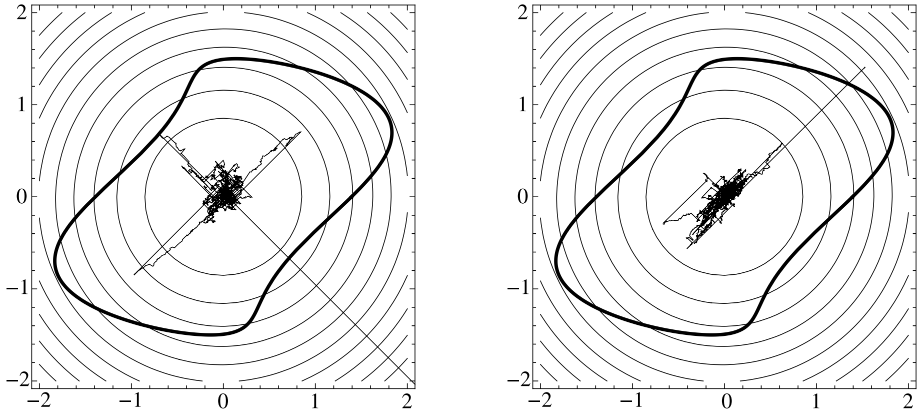

Fig. 1 Simulated trajectories of the jump diffusion X^ε driven by two 1.25-stable processes L^1 and L^2 (l.) and by a 1.25-stable processes L^1 and a 1.75-stable process L^2 (r.) in directions $r_1 = (\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ and $r_2 = (-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}})$ respectively. In the secons case, the driver L^1 with the smaller stability index determines the exit

For $i \in \{1, \ldots, N\}$ we define

$$\begin{aligned} p_s &:= \sum_{k=1}^{N} \lambda_k^{\alpha_1}((d_i^+)^{-\alpha_1} + (-d_i^-)^{-\alpha_1}), \\ p_i^+ &:= \lambda_i^{\alpha_1} \cdot (d_i^+)^{-\alpha_1}, \\ p_i^- &:= \lambda_i^{\alpha_1} \cdot (-d_i^-)^{-\alpha_1}, \end{aligned} \tag{3.7}$$

and conclude that for every $\delta \in (0, \delta_0)$ and initial state $x \in \mathcal{G}$

$$\begin{aligned} \lim_{\varepsilon \to 0} \mathbf{P}(X^\varepsilon_{\sigma(\varepsilon)} \in \Omega_i^+(\delta)) &= \frac{p_i^+}{p_s}, \\ \lim_{\varepsilon \to 0} \mathbf{P}(X^\varepsilon_{\sigma(\varepsilon)} \in \Omega_i^-(\delta)) &= \frac{p_i^-}{p_s}, \end{aligned} \tag{3.8}$$

for $i = 1, \ldots, N$ and

$$\sum_{i=1}^{N} \frac{p_i^+ + p_i^-}{p_s} = 1. \tag{3.9}$$

Therefore, X^ε exits from the domain $\mathcal{G}$ in a little tube in the directions of the smallest α (see Fig. 1). In the remaining sections of this paper we will make the above heuristic arguments rigorous. We proceed in four steps.

First step: We prove that as a consequence of exponential stability (C1) the waiting time for a large jump is longer than the time for X^ε to enter a small δ_0-ball around zero. If the starting point x has a certain distance to the boundary of $\mathcal{G}$ and keeps this distance such that small jumps can be ignored, X^ε will leave the domain with a large jump near zero (compare (C4)).

Second step: Due to the strong Markov property of X^ε, we will see that it is enough to do step one at time τ_1^*.

Third step: Since the distance functions to the boundary of $\mathcal{G}$ are smooth enough we can estimate a jump from a place close to zero by a jump from zero. So the probability of exiting in direction i will be seen to be proportional to $\nu_i(\mathbb{R}\backslash[d_i^-, d_i^+])$, i.e. to correspond to a large jump from zero out of the domain.

Fourth step: For small ε the processes with α_i-stable jump component in direction i with the smallest α_i are dominating. Therefore, as we will see, exit time and exit place are determined by the sum of the $\nu_i(\mathbb{R}\backslash[d_i^-, d_i^+])$ of the first N directions (compare (C3)).

4 The Main Result: Exit from the Domain $\mathcal{G}$

Let us now complement the heuristic arguments of the preceding section with rigorous proofs. We state the main theorems of this paper.

For $\varepsilon > 0$, $\gamma > 0$ we define the following interior parts $\mathcal{G}_k$ of the domain $\mathcal{G}$:

$$\mathcal{G}_k := \mathcal{G}_k(\gamma, \varepsilon) = \{x \in \mathcal{G} : \operatorname{dist}(x, \partial\mathcal{G}) \geq (k + C)\varepsilon^\gamma\}, \quad k = 1, 2, 3, 4. \tag{4.1}$$

We consider the exit time σ_ε from the set $\mathcal{G}_1$, defined by

$$\sigma_\varepsilon := \sigma_\varepsilon(\gamma, x_0) = \inf\{t \geq 0 : X_t^\varepsilon(x_0) \notin \mathcal{G}_1\}, \tag{4.2}$$

for $x_0 \in \mathcal{G}_1$.

For $0 < \delta < \delta_0$ and $i \in \{1, \ldots, M + N\}$ we also define the small tubes in direction $\pm r_i$

$$\begin{aligned} \Omega_i^{\varepsilon+}(\delta) &:= \{y \in \mathbb{R}^d : \exists z \in K(\delta), t > d_i^+(z) - (1 + C_r)\varepsilon^\gamma : y = t \cdot r_i + z\}, \\ \Omega_i^{\varepsilon-}(\delta) &:= \{y \in \mathbb{R}^d : \exists z \in K(\delta), t < d_i^-(z) + (1 + C_r)\varepsilon^\gamma : y = t \cdot r_i + z\}. \end{aligned} \tag{4.3}$$

In these terms we prove the following theorem.

Theorem 1 *There exists a constant $\gamma_0 > 0$, such that for $\gamma \in (0, \gamma_0)$, $\theta > -1$ and starting points $x_0 \in \mathcal{G}$ the following holds true*:

$$\lim_{\varepsilon \downarrow 0} \mathbf{E} e^{-\theta m_\varepsilon \sigma_\varepsilon} = \frac{1}{\theta + 1} \tag{4.4}$$

and thus for any $t \geq 0$

$$\lim_{\varepsilon \downarrow 0} \mathbf{P}(m_\varepsilon \sigma_\varepsilon > t) \to e^{-t} \tag{4.5}$$

with m_ε defined in (2.13), *that is in the small noise limit the normalized exit time $m_\varepsilon \sigma_\varepsilon$ is exponentially distributed with mean 1. Moreover, for any $p > 0$ we have $\mathbf{E}(m_\varepsilon \sigma_\varepsilon)^p \to \int_0^\infty y^p e^{-y}\, dy$. Furthermore, for every $\delta \in (0, \delta_0)$ the probabilities to exit in direction $\pm r_i$ are given by*

$$\begin{aligned} \lim_{\varepsilon \downarrow 0} \mathbf{P}(X_{\sigma_\varepsilon}^\varepsilon \in \Omega_i^{\varepsilon+}(\delta)) &= \frac{p_i^+}{p_s}, \\ \lim_{\varepsilon \downarrow 0} \mathbf{P}(X_{\sigma_\varepsilon}^\varepsilon \in \Omega_i^{\varepsilon-}(\delta)) &= \frac{p_i^-}{p_s}, \end{aligned} \tag{4.6}$$

for $x \in \mathcal{G}_2$ and $1 \leq i \leq N$. In particular we have

$$\sum_{i=1}^{N} \frac{1}{p_s}(p_i^+ + p_i^-) = 1. \tag{4.7}$$

For technical reasons we formulate the following two propositions for the Laplace transform of the normalized exit time. Theorem 1 is their immediate consequence.

Proposition 1 *For every $\theta > -1$ and $K > 0$ there exists $\varepsilon_0 > 0$, such that for every $\varepsilon \in (0, \varepsilon_0]$:*

$$\mathbf{E}e^{-\theta m_\varepsilon \sigma_\varepsilon} \geq \frac{1-K}{1+\theta+K}, \tag{4.8}$$

for starting points $x \in \mathcal{G}_2$.

Proposition 2 *For every $\theta > -1$ and $K \in (0, \theta + 1)$ there exists $\varepsilon_0 > 0$, such that for every $\varepsilon \in (0, \varepsilon_0]$*

$$\mathbf{E}e^{-\theta m_\varepsilon \sigma_\varepsilon} \leq \frac{1+K}{1+\theta-K}, \tag{4.9}$$

for starting points $x \in \mathcal{G}_2$.

In the proofs, it will be convenient to decompose the process X^ε into a part driven by the small jump component alone, and a complementary large jump part. For this purpose, for $t \geq 0$, $\varepsilon > 0$ and $x_0 \in \mathcal{G}$ we define the small jump part by

$$l_t := \sum_{i=1}^{N+M} \lambda_i r_i \xi_t^i, \tag{4.10}$$

and the corresponding small jump component of X^ε by

$$y_t(x_0) := x_0 + \int_0^t b(y_s)ds + \varepsilon l_t, \quad t \geq 0. \tag{4.11}$$

Recall the constant C, δ_0 and T_0 from the assumptions. Fixing $q := 2 + C^2$ and $\gamma \in (0, \frac{1}{q})$, for $n = \min(\frac{1}{3C}, \frac{1}{2})$ we choose ε small enough, such that $\frac{n}{2}\varepsilon^\gamma < \delta_0$. In these terms we define the relaxation time

$$T_R := \sup_{\|x\| \leq \delta_0} \inf\left\{t > 0 : \|z_t(x)\| \leq \frac{n}{2}\varepsilon^\gamma\right\} + T_0. \tag{4.12}$$

Furthermore we consider the following processes, which are defined for $t \in [0, S_k^*]$, $k \in \mathbb{N}$, and $i = 1, \ldots, N + M$:

$$\xi_t^{i,k} := \xi^i_{t+\tau^*_{k-1}} - \xi^i_{\tau^*_{k-1}}. \tag{4.13}$$

$\xi_t^{i,k}$ is a Lévy process and due to the strong Markov property we know that $\xi^{i,k}$ has the same law as ξ^i. Additionally we have:

$$v_t^k(x) := x + \int_0^t b(v_s^k)ds + \sum_{i=1}^{N+M} \varepsilon \lambda_i r_i \xi_t^{i,k} \quad \text{for } t \in [0, S_k^*], \tag{4.14}$$

$$l_t^k := \sum_{i=1}^{N+M} \lambda_i r_i \xi_t^{i,k}, \tag{4.15}$$

for $x \in \mathcal{G}$ and $k \in \mathbb{N}$. We have $v_t^1(x) = x + \int_0^t b(v_s^1)ds + \varepsilon l_t = y_t(x)$ for $t \in [0, \tau_1^*]$. For $k \geq 1$ the jumps at times τ_k^* are described by

$$W_1^* = \sum_{i=1}^{N+M} r_i \lambda_i W_1^i \mathbb{I}\{\tau_1^* = \tau_1^i\}, \tag{4.16}$$

$$W_k^* = \sum_{i=1}^{N+M} \sum_{j=1}^{k} r_i \lambda_i W_j^i \mathbb{I}\{\tau_k^* = \tau_j^i\}. \tag{4.17}$$

The set $\{(W_k^*)_{k\in\mathbb{N}}, (S_k^*)_{k\in\mathbb{N}}\}$ is independent. In this notation, for $x \in \mathcal{G}$ the process X^ε can be further specified by

$$\begin{aligned} X_t^\varepsilon(x) &= v_t^1(x) + \varepsilon W_1^* \mathbb{I}\{t = \tau_1^*\}, \quad t \in [0, S_1^*], \\ X_{t+\tau_1^*}^\varepsilon(x) &= v_t^2(v_{\tau_1^*}^1(x) + \varepsilon W_1^*) + \varepsilon W_2^* \mathbb{I}\{t = S_2^*\}, \quad t \in [0, S_2^*], \\ X_{t+\tau_{k-1}^*}^\varepsilon(x) &= v_t^k(v_{\tau_{k-1}^*}^{k-1} + \varepsilon W_{k-1}^*) + \varepsilon W_k^* \mathbb{I}\{t = S_k^*\}, \quad t \in [0, S_k^*]. \end{aligned} \tag{4.18}$$

For the processes v^k, $k \in \mathbb{N}$, and $y \in \mathcal{G}$ we define

$$\begin{aligned} A^k(y) &:= \{v_t^k(y) \in \mathcal{G}_1, t \in [0, S_k^*), v_{\tau_k^*}^k(y) + \varepsilon W_k^* \in \mathcal{G}_1\}, \\ B^k(y) &:= \{v_t^k(y) \in \mathcal{G}_1, t \in [0, S_k^*), v_{\tau_k^*}^k(y) + \varepsilon W_k^* \notin \mathcal{G}_1\}, \\ A_k^-(y) &:= \{v_t^k(y) \in \mathcal{G}_1, t \in [0, S_k^*), v_{\tau_k^*}^k(y) + \varepsilon W_k^* \in \mathcal{G}_2\}, \\ \tilde{A}^k(y) &:= \{v_t^k(y) \in \mathcal{G}_1, t \in [0, S_k^*), v_{\tau_k^*}^k(y) + \varepsilon W_k^* \in \mathcal{G}_1 \backslash \mathcal{G}_2\}. \end{aligned} \tag{4.19}$$

The analogous sets in the directions $i \in \{1, \dots, M + N\}$ are correspondingly given by

$$\begin{aligned} A^{1,i}(y) &:= \{v_t^1(y) \in \mathcal{G}_1, t \in [0, \tau_1^i), v_{\tau_1^i}^1(y) + \varepsilon \lambda_i r_i W_1^i \in \mathcal{G}_1\}, \\ B^{1,i}(y) &:= \{v_t^1(y) \in \mathcal{G}_1, t \in [0, \tau_1^i), v_{\tau_1^i}^1(y) + \varepsilon \lambda_i r_i W_1^i \notin \mathcal{G}_1\}, \\ A_{1,i}^-(y) &:= \{v_t^1(y) \in \mathcal{G}_1, t \in [0, S_1^i), v_{\tau_1^i}^1(y) + \varepsilon \lambda_i r_i W_1^i \in \mathcal{G}_2\}, \\ \tilde{A}^{1,i}(y) &:= \{v_t^1(y) \in \mathcal{G}_1, t \in [0, S_1^i), v_{\tau_1^i}^1(y) + \varepsilon \lambda_i r_i W_1^i \in \mathcal{G}_1 \backslash \mathcal{G}_2\}. \end{aligned} \tag{4.20}$$

5 Some Preliminary Estimates

We fix $q := 2 + C^2$ and $\gamma \in (0, \frac{1}{q})$ with C from the assumptions. Let

$$\mathcal{E}_T(\epsilon, \gamma) = \left\{\omega \in \Omega : \sup_{t\in[0,T]} \|\varepsilon l_t(\omega)\| \leq \varepsilon^{q\gamma}\right\}. \tag{5.1}$$

Lemma 1 *Let $T > 0$ and $\gamma \in (0, \frac{1}{q})$. There exists $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$ we have*

$$\sup_{t\in[0,T_R]} \|y_t - z_t\| \leq \frac{1}{2}\varepsilon^\gamma, \tag{5.2}$$

on $\mathcal{E}_T(\epsilon, \gamma)$, for $T_R < T$.

Proof Let $x_0 \in \mathcal{G}$, $\omega \in \mathcal{E}_T(\epsilon, \gamma)$, $(y_t)_{t\geq 0} = (y_t(x_0)(\omega))_{t\geq 0}$ and $l = l(\omega)$. Since b is Lipschitz continuous, we have for $t \geq 0$

$$\|y_t - z_t\| \leq C\int_0^t \|y_s - z_s\| ds + \|\varepsilon l_t\|. \tag{5.3}$$

With the help of Gronwall's lemma we conclude

$$\|y_t - z_t\| \leq e^{Ct}\|\varepsilon l_t\|, \tag{5.4}$$

and therefore on $\mathcal{E}_T(\epsilon, \gamma)$ and by $T_R \leq T$:

$$\sup_{t\in[0,T_R]} \|y_t - z_t\| \leq e^{C\cdot T_R} \sup_{t\in[0,T_R]} \|\varepsilon l_t\| \leq e^{C(C\gamma|\ln\varepsilon|+T_0)}\varepsilon^{q\gamma}, \tag{5.5}$$

since $T_R \leq C\gamma|\ln\varepsilon| + T_0$ because of the exponential stability. With the choice of q, there exists an ε_0 such that for all $\varepsilon \in (0, \varepsilon_0)$:

$$\sup_{t\in[0,T_R]} \|y_t - z_t\| \leq \frac{\varepsilon^\gamma}{2}, \tag{5.6}$$

for $T_R \leq T$ and on $\mathcal{E}_T(\epsilon, \gamma)$. □

Let us next define

$$y_t^k(x) = x + \int_k^t b(y_s^k)ds + \varepsilon(l_t - l_k), \tag{5.7}$$

for $k \in \mathbb{R}$, $k \leq t$. Then we have $y_t^k(y_k(x)) = y_t(x)$ and thus

$$\|y_t^k(x) - z_{t-k}(x)\| = \left\|\int_k^t (b(y_s^k) - b(z_s))ds + \varepsilon(l_t - l_k)\right\|. \tag{5.8}$$

By an analogous estimation as in the proof of the previous lemma we find

$$\sup_{t\in[k,k+m]} \|y_t^k(x) - z_{t-k}(x)\| \leq \frac{\varepsilon^\gamma}{2}, \tag{5.9}$$

on $\{\sup_{t\in[k,k+m]} \|\varepsilon(l_t - l_k)\| \leq \varepsilon^{q\gamma}\}$, for $m > 0$. Let $\gamma > 0$ and $\varepsilon_0 > 0$ be chosen so that this holds true.

Lemma 2 *Let T_ε be a non-negative random variable. For all $k \in \mathbb{N}$, $r > 0$, there exists $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$*

$$\left\{\sup_{t\in[0,T_\varepsilon]}\|y_t(x)-z_t(x)\|\geq\frac{\varepsilon^\gamma}{2}\right\}$$
$$\subseteq\left\{T_\varepsilon\geq\frac{k}{\varepsilon^r}\right\}\cup\bigcup_{j=0}^{k-1}\left\{\sup_{t\in[\frac{j}{\varepsilon^r},\frac{j+1}{\varepsilon^r}]}\|\varepsilon(l_t-l_{\frac{j}{\varepsilon^r}})\|\geq\varepsilon^{q\gamma}\right\}. \tag{5.10}$$

Proof Let $x\in\mathcal{G}_1$ and $r>0$. We choose $\varepsilon_0>0$ such that for $0<\varepsilon<\varepsilon_0$ we have $T_R<\frac{1}{\varepsilon^r}$. Then we may decompose

$$\left\{\sup_{t\in[0,T_\varepsilon]}\|y_t(x)-z_t(x)\|\geq\frac{\varepsilon^\gamma}{2}\right\}$$
$$\subseteq\left\{T_\varepsilon\geq\frac{k}{\varepsilon^r}\right\}\cup\left\{\sup_{t\in[0,\frac{k}{\varepsilon^r}]}\|y_t(x)-z_t(x)\|\geq\frac{\varepsilon^\gamma}{2}\right\}. \tag{5.11}$$

With the definition of T_R we have $\|z_{\frac{1}{\varepsilon^r}}(x)\|\leq\frac{\varepsilon^\gamma}{2}$. On $\{\sup_{t\in[0,\frac{1}{\varepsilon^r}]}\|y_t(x)-z_t(x)\|\leq\frac{\varepsilon^\gamma}{2}\}$ we obtain $\|y_{\frac{1}{\varepsilon^r}}(x)\|\leq\varepsilon^\gamma$ and therefore,

$$\left\{\sup_{t\in[0,\frac{k}{\varepsilon^r}]}\|y_t(x)-z_t(x)\|\geq\frac{\varepsilon^\gamma}{2}\right\}$$
$$\subseteq\bigcup_{l=0}^{k-1}\left[\bigcap_{j=1}^{l-1}\left\{\sup_{t\in[\frac{j-1}{\varepsilon^r},\frac{j}{\varepsilon^r}]}\|y_t(x)-z_{t-\frac{j-1}{\varepsilon^r}}(y_{\frac{j-1}{\varepsilon^r}}(x))\|<\frac{\varepsilon^\gamma}{2}\right\}\right.$$
$$\left.\cap\left\{\sup_{t\in[\frac{l}{\varepsilon^r},\frac{l+1}{\varepsilon^r}]}\|y_t(x)-z_{t-\frac{l}{\varepsilon^r}}(y_{\frac{l}{\varepsilon^r}}(x))\|\geq\frac{\varepsilon^\gamma}{2}\right\}\right]$$
$$\subseteq\bigcup_{l=0}^{k-1}\left[\bigcap_{j=1}^{l-1}\{\|y_{\frac{j}{\varepsilon^r}}(x)\|\leq\varepsilon^\gamma\}\right.$$
$$\left.\cap\left\{\sup_{t\in[\frac{l}{\varepsilon^r},\frac{l+1}{\varepsilon^r}]}\|y_t^{\frac{l}{\varepsilon^r}}(y_{\frac{l}{\varepsilon^r}}(x))-z_{t-\frac{l}{\varepsilon^r}}(y_{\frac{l}{\varepsilon^r}}(x))\|\geq\frac{\varepsilon^\gamma}{2}\right\}\right]$$
$$\subseteq\left\{\sup_{t\in[0,\frac{1}{\varepsilon^r}]}\|y_t(x)-z_t(x)\|\geq\frac{\varepsilon^\gamma}{2}\right\}$$
$$\cup\bigcup_{l=1}^{k-1}\left\{\sup_{t\in[\frac{l}{\varepsilon^r},\frac{l+1}{\varepsilon^r}]}\|y_t^{\frac{l}{\varepsilon^r}}(a)-z_{t-\frac{l}{\varepsilon^r}}(a)\|\geq\frac{\varepsilon^\gamma}{2},\text{ for some }a\in\mathbb{R},\ \|a\|\leq\varepsilon^\gamma\right\}$$
$$\subseteq\bigcup_{j=0}^{k-1}\left\{\sup_{t\in[\frac{j}{\varepsilon^r},\frac{j+1}{\varepsilon^r}]}\|\varepsilon(l_t-l_{\frac{j}{\varepsilon^r}})\|\geq\varepsilon^{\gamma q}\right\}. \tag{5.12}$$

This completes the proof. □

Lemma 3 *Let $\rho\in(0,1)$ and $\gamma\in(0,\frac{1-\rho}{q})$. Let $\varepsilon\mapsto T_\varepsilon$ be a positive function with $h_\varepsilon:=T_\varepsilon^{-1}\varepsilon^{q\gamma+\rho-1}\to\infty$ with $\varepsilon\to0$. There exists $p_0>0$ and $\varepsilon_0>0$ such that for all $0<\varepsilon<\varepsilon_0$*

and $0 < p < p_0$:

$$\mathbf{P}\Big(\sup_{t\in[0,T_\varepsilon]} \|\varepsilon l_t\| \geq \varepsilon^{q\gamma}\Big) \leq \exp(-\varepsilon^{-p}). \tag{5.13}$$

Proof Since the $\varepsilon\xi^i$ are symmetrical mean zero martingales with compact support, we can use Doob's inequality for exponential functions of martingales to derive for $\varepsilon > 0, u_\varepsilon > 0$

$$\mathbf{P}\Big(\sup_{t\in[0,T_\varepsilon]} \|\varepsilon l_t\| \geq \varepsilon^{q\gamma}\Big) \leq 2\cdot e^{-u_\varepsilon \varepsilon^{q\gamma}} \prod_{i=1}^{N+M} \sup_{t\in[0,T_\varepsilon]} \mathbf{E} e^{u_\varepsilon \varepsilon \lambda_i \xi_t^i}. \tag{5.14}$$

The last part of this expression is known from the Lévy-Khintchine representation. Hence we have the following chain of inequalities:

$$\begin{aligned}
&\sup_{t\in[0,T_\varepsilon]} \mathbf{E}[e^{u_\varepsilon \varepsilon \lambda_i \xi_t^i}] \\
&\quad = \sup_{t\in[0,T_\varepsilon]} \exp\bigg((u_\varepsilon \varepsilon \lambda_i)^2 t\frac{b}{2} + t\int_{|y|\leq \frac{1}{\lambda_i \varepsilon^\rho}} (e^{u_\varepsilon \varepsilon \lambda_i y} - 1 - u_\varepsilon \varepsilon \lambda_i y \mathbb{I}\{|y|<1\})\nu(dy)\bigg) \\
&\quad \leq \sup_{t\in[0,T_\varepsilon]} \exp\bigg((u_\varepsilon \varepsilon \lambda_i)^2 t\frac{b}{2} + t\int_{|y|\leq \frac{1}{\lambda_i \varepsilon^\rho}} (1\wedge y^2)\exp(u_\varepsilon \varepsilon^{1-\rho})\nu(dy)\bigg) \\
&\quad \leq \exp\bigg(T_\varepsilon\bigg((u_\varepsilon \varepsilon \lambda_i)^2 \frac{b}{2} + m\exp(u_\varepsilon \varepsilon^{1-\rho})\bigg)\bigg),
\end{aligned} \tag{5.15}$$

where we set $m = \int_{\mathbf{R}} (1\wedge y^2)\nu(dy) < \infty$. With $h(\varepsilon) := \frac{\varepsilon^{q\gamma+\rho-1}}{T_\varepsilon}$, $u(\varepsilon) := \varepsilon^{\rho-1}\ln h(\varepsilon)$ and $n := \sum_{i=1}^{N+M} \lambda_i^2$ we see

$$\begin{aligned}
\prod_{i=1}^{N+M} \sup_{t\in[0,T_\varepsilon]} \mathbf{E}[\exp(u_\varepsilon \varepsilon \lambda_i \xi_t^i)] &\leq \exp\Big(T_\varepsilon \varepsilon^{2\rho}(\ln h(\varepsilon))^2 n\frac{b}{2} + m(N+M)T_\varepsilon h(\varepsilon)\Big) \\
&\leq \exp(2T_\varepsilon m h(\varepsilon)(N+M)) \\
&\leq \exp(2\varepsilon^{\rho+q\gamma-1}(N+M)m),
\end{aligned} \tag{5.16}$$

for ε small enough that $n\frac{b}{2}(\ln h(\varepsilon))^2 \leq m h(\varepsilon)$. For $p_0 := -\frac{q\gamma+\rho-1}{2}$ and $0 < p < p_0$ we have

$$\mathbf{P}\Big(\sup_{t\in[0,T_\varepsilon]} \|\varepsilon l_t\| \geq \varepsilon^{q\gamma}\Big) \leq \exp(-\varepsilon^{-p}) \tag{5.17}$$

for ε small enough. This finishes the proof. □

For $x \in \mathcal{G}$, $n \geq 1$ let now

$$\begin{aligned}
E_x^n &:= \bigg\{\omega \in \Omega : \sup_{t\in[0,S_n^*]} \|y_t^n(x)(\omega) - z_t(x)(\omega)\| \leq \frac{\varepsilon^\gamma}{2}\bigg\}, \\
E &:= E_x^1.
\end{aligned} \tag{5.18}$$

With Lemma 2 we see:

$$(E_x^n)^c \subseteq \bigg\{S_n^* \geq \frac{k}{\varepsilon^q}\bigg\} \cup \bigcup_{j=0}^{k-1} \bigg\{\sup_{t\in[\frac{j}{\varepsilon^r},\frac{j+1}{\varepsilon^r}]} \|\varepsilon(l_t^n - l_{\frac{j}{\varepsilon^r}}^n)\| \geq \varepsilon^{q\gamma}\bigg\}. \tag{5.19}$$

For $0 < r < \min(1-\rho-q\gamma, \alpha_1\rho)$ and $k_\varepsilon = [\frac{\varepsilon^{r/2}}{\beta^S}]$, where $[\cdot]$ denotes the integer part, we can prove the following Lemma.

Lemma 4 *For every $\theta > -1$ there exists a $p > 0$ and an $\varepsilon_0 > 0$, such that for every $\varepsilon < \varepsilon_0$:*

$$\mathbf{E}[e^{-\theta m_\varepsilon \tau_1^*}\mathbb{I}\{E^c\}] \le \frac{\beta^S}{\beta^S+\theta m_\varepsilon} e^{-\varepsilon^{-p}}. \tag{5.20}$$

Proof Since $r < 1-\rho-q\gamma$, with the help of Lemma 3, the Markov property of the ξ^i and the independence of τ_1^* and ξ^i, we see that there exists $\varepsilon_1 > 0$ and $p' > 0$, such that for $0 < \varepsilon < \varepsilon_1$, $0 \le j \le k_\varepsilon - 1$:

$$\mathbf{E}\left[e^{-\theta m_\varepsilon \tau_1^*}\mathbb{I}\left\{\sup_{t\in[\frac{j}{\varepsilon^r},\frac{j+1}{\varepsilon^r}]} \|\varepsilon(l_t^1 - l_{\frac{j}{\varepsilon^r}}^1)\| \ge \varepsilon^{q\gamma}\right\}\right] \le \frac{\beta^S}{\beta^S+\theta m_\varepsilon} e^{-\varepsilon^{-p'}}. \tag{5.21}$$

With (5.19) we obtain for an $0 < \varepsilon_0 \le \varepsilon_1$, such that for $0 < \varepsilon \le \varepsilon_0$ we have $\beta^S + \theta m_\varepsilon > 0$:

$$\begin{aligned}
&\mathbf{E}[e^{-\theta m_\varepsilon \tau_1^*}\mathbb{I}\{E^c\}] \\
&\quad \le \mathbf{E}\left[e^{-\theta m_\varepsilon \tau_1^*}\mathbb{I}\left\{\tau_1^* \ge \frac{k_\varepsilon}{\varepsilon^r}\right\}\right] \\
&\qquad + \sum_{j=0}^{k_\varepsilon-1} \mathbf{E}\left[e^{-\theta m_\varepsilon \tau_1^*}\mathbb{I}\left\{\sup_{t\in[\frac{j}{\varepsilon^r},\frac{j+1}{\varepsilon^r}]} \|\varepsilon(l_t^1 - l_{\frac{j}{\varepsilon^r}}^1)\| \ge \varepsilon^{q\gamma}\right\}\right] \\
&\quad \le \int_{k_\varepsilon/\varepsilon^r}^\infty \beta^S e^{-(\theta m_\varepsilon+\beta^S)t}dt + k_\varepsilon \cdot \frac{\beta^S}{\beta^S+\theta m_\varepsilon} e^{-\varepsilon^{-p'}} \\
&\quad \le \frac{\beta^S}{\beta^S+\theta m_\varepsilon}(e^{-(\theta m_\varepsilon+\beta^S)k_\varepsilon\varepsilon^{-r}} + k_\varepsilon e^{-\varepsilon^{-p'}}).
\end{aligned} \tag{5.22}$$

Since $k_\varepsilon = O(\varepsilon^{r/2-\alpha_1\rho})$ and $\beta^S k_\varepsilon \varepsilon^{-r} = O(\varepsilon^{-r/2})$, there exists a $p > 0$ such that for all $0 < \varepsilon < \varepsilon_0$ the last estimate can be bounded above by

$$\frac{\beta^S}{\beta^S+\theta m_\varepsilon} e^{-\varepsilon^{-p}}. \tag{5.23}$$

□

Lemma 5 *Let $y \in \mathcal{G}_2$ and $\gamma \in (0, \frac{1}{q})$. There exists an $\varepsilon_0 > 0$ s.t. for $\varepsilon \in (0, \varepsilon_0)$, $1 \le i \le M+N$:*

1. $\mathbb{I}\{A_{1,i}^-(y)\} \ge \mathbb{I}\{\varepsilon\lambda_i W_1^i \in (d_i^- + (2+2C)\varepsilon^\gamma, d_i^+ - (2+2C)\varepsilon^\gamma)\}$,
2. $\mathbb{I}\{B^{1,i}(y)\} \ge \mathbb{I}\{\varepsilon\lambda_i W_1^i \notin (d_i^-, d_i^+)\}$,

on $E \cap \{T_R < \tau_1^i\}$.

Proof If $\varepsilon_0^\gamma < \delta_0$, we can easily check the estimate taking into account that on the events E and $\{T_R < \tau_1^i\}$ we have $\|y_{\tau_1^i}\| \le \varepsilon^\gamma$. Remember $C > 1$. □

Lemma 6 *Let $\gamma \in (0, \frac{1}{q})$. For $y \in \mathcal{G}$, $1 \le i \le M+N$, there is an $\varepsilon_0 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$*

$$A_{1,i}^-(y) \supseteq E \cap \{|\varepsilon\lambda_i W_1^i| < \varepsilon^\gamma\} \cap \{5C^2\varepsilon^\gamma \le \tau_1^i\}. \tag{5.24}$$

Proof If we choose $\varepsilon_0 > 0$ small enough such that $(4+C)\varepsilon_0^\gamma \le \delta_0$, we can conclude for $\varepsilon < \varepsilon_0$ with the help of (C6)

$$\sup_{x\in\mathcal{G}\setminus\mathcal{G}_4} \inf\{t>0 : z_t(x) \in \mathcal{G}_4\} \le 5C^2\varepsilon^\gamma, \tag{5.25}$$

remembering $C > 1$. I.e. starting in $\mathcal{G}_2$ we know $z_t \in \mathcal{G}_4$ on $\tau_1^i > t \ge 5C^2\varepsilon^\gamma$ and therefore on E: $y_t \in \mathcal{G}_3 \subset \mathcal{G}_2$. □

The next lemma is proven with the same arguments as the preceding ones.

Lemma 7 *Let* $i \in \{1,\dots,M+N\}$, $\gamma \in (0,\frac{1}{q})$ *and* $y \in \mathcal{G}_1$. *There is* $\varepsilon_0 > 0$, *such that for all* $\varepsilon \in (0,\varepsilon_0)$:

1. $\mathbb{I}\{A^{1,i}(y)\} \le \mathbb{I}\{\varepsilon\lambda_i W_1^i \in [d_i^-, d_i^+]\}$,
2. $\mathbb{I}\{B^{1,i}(y)\} \le \mathbb{I}\{\varepsilon\lambda_i W_1^i \notin [d_i^- + (1+2C)\varepsilon^\gamma, d_i^+ - (1+2C)\varepsilon^\gamma]\}$,
3. $\mathbb{I}\{\tilde{A}^{1,i}(y)\} \le \mathbb{I}\{\varepsilon\lambda_i W_1^i \in [d_i^-, d_i^- + (2+2C)\varepsilon^\gamma]\}$
   $+\, \mathbb{I}\{\varepsilon\lambda_i W_1^i \in [d_i^+ - (2+2C)\varepsilon^\gamma, d_i^+]\}$,

on $E \cap \{\tau_1^i \ge T_R\}$. □

Proposition 3 *Let* $\rho \in (0,1)$ *and* $\gamma \in (0, \frac{1-\rho}{q})$. *For* $\theta > -1$ *and* $m > 0$ *there exist constants* $p, C_1, C_2, \varepsilon_0 > 0$, *such that for* $\varepsilon \in (0,\varepsilon_0)$:

$$\begin{aligned}
1.\quad & \sum_{i=1}^{N+M} \mathbf{E}[e^{-\theta m_\varepsilon \tau_1^*}\mathbb{I}\{\tau_1^i = \tau_1^*\}\mathbb{I}\{\tau_1^* < T_R\}\mathbb{I}\{|\varepsilon\lambda_i W_1^i| \ge \varepsilon^\gamma\}] \\
& \quad \le C_1 \varepsilon^{\alpha_1(1-\rho-\gamma)} \frac{(\beta^S)^2|\ln(\varepsilon)|}{\theta m_\varepsilon + \beta^S}, \\
2.\quad & \sum_{i=1}^{N+M} \mathbf{E}[e^{-\theta m_\varepsilon \tau_1^*}\mathbb{I}\{\tau_1^i = \tau_1^*\}\mathbb{I}\{E^c\}] \le \frac{\beta^S}{\beta^S + \theta m_\varepsilon}\exp(-\varepsilon^{-p}), \\
3.\quad & \sum_{i=1}^{N+M} \mathbf{E}[e^{-\theta m_\varepsilon \tau_1^*}\mathbb{I}\{\tau_1^i = \tau_1^*\}\mathbb{I}\{\tau_1^i < m\varepsilon^\gamma\}] \le C_2 \frac{\beta^S}{\beta^S + \theta m_\varepsilon}\beta^S\varepsilon^\gamma.
\end{aligned} \tag{5.26}$$

Proof A straightforward calculation yields that for two independent exponentially distributed r.v. X, Y with parameters $\alpha > 0$ resp. $\beta > 0$

$$\mathbf{E}[e^{-\theta X}\mathbb{I}\{X \le Y\}\mathbb{I}\{X < k\}) = \frac{\alpha}{\alpha+\beta+\theta}(1 - e^{-(\alpha+\beta+\theta)k}), \tag{5.27}$$

for $k > 0$ and $\theta > \min(-\alpha, -\beta, -(\alpha+\beta))$. Additionally, exponential stability and the definition of T_R imply that there is $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$ and a constant $C_1 > 0$:

$$(\beta^S + \theta m_\varepsilon)T_R \le (\beta^S + \theta m_\varepsilon)(C|\ln\varepsilon| + T_0) \le C_1\beta^S|\ln\varepsilon|. \tag{5.28}$$

We use this together with the independence and the law properties of τ_1^i and W_1^i to prove the proposition. □

Let $I_i := [d_i^- + (2+2C)\varepsilon^\gamma, d_i^+ - (2+2C)\varepsilon^\gamma]$. The following two propositions are proved as the preceding one, taking into account the properties of τ_1^i and W_1^i and the following argument. Let ε be small enough such that $\min(d_i^+, -d_i^-) > \varepsilon^{1-\rho}$ and for a fixed $K > 0$: $K\varepsilon^\gamma < \min(-d_i^-, d_i^+)$. Then we estimate

$$\mathbf{P}[\varepsilon\lambda_i W_1^i \in [d_i^- + \varepsilon^\gamma K, d_i^+ - \varepsilon^\gamma K]] \tag{5.29}$$

$$= 1 - \frac{1}{\beta_i\alpha_i}(\lambda_i\varepsilon)^{\alpha_i}[(-d_i^- - \varepsilon^\gamma K)^{-\alpha_i} + (d_i^+ - \varepsilon^\gamma K)^{-\alpha_i}] \tag{5.30}$$

$$\geq 1 - \frac{m_i}{\beta_i}(1 + \kappa\varepsilon^l), \tag{5.31}$$

for a $\kappa > 0$, $l > 0$ and $i \in \{1, \dots, M + N\}$.

Proposition 4 *Let $\rho \in (0, 1)$ and $\gamma \in (0, \frac{1-\rho}{q})$. For $\theta > -1$ there are constants ε_0, l, C_3, $C_4 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0]$:*

$$\begin{aligned}
&1. \quad \sum_{i=1}^{N+M} \mathbf{E}[e^{-\theta m_\varepsilon \tau_1^i}\mathbb{I}\{\tau_1^i = \tau_1^*\}\mathbb{I}\{\varepsilon W_1^i\lambda_i \in I_i\}] \\
&\qquad \geq \frac{\beta^S}{\beta^S + m_\varepsilon\theta}\left(1 - \frac{m_\varepsilon}{\beta^S}(1 + C_3\varepsilon^l)\right), \\
&2. \quad \sum_{i=1}^{N+M} \mathbf{E}[e^{-\theta m_\varepsilon \tau_1^i}\mathbb{I}\{\tau_1^i > T_R\}\mathbb{I}\{\tau_1^i = \tau_1^*\}\mathbb{I}\{\varepsilon\lambda_i W_1^i \notin [d_i^-, d_i^+]\}] \\
&\qquad \geq \frac{\beta^S}{m_\varepsilon\theta + \beta^S}\frac{m_\varepsilon}{\beta^S}(1 - C_4\beta^S|\ln\varepsilon|).
\end{aligned} \tag{5.32}$$

Proposition 5 *Let $\rho \in (0, 1)$ and $\gamma \in (0, \frac{1-\rho}{q})$. For $\theta > -1$ there exits $\varepsilon_0 > 0$ and $C_5 > 0$, such that for all $\varepsilon \in (0, \varepsilon_0)$:*

$$\begin{aligned}
&1. \quad \sum_{i=1}^{N+M} \mathbf{E}[e^{-\theta m_\varepsilon \tau_1^i}\mathbb{I}\{\tau_1^i = \tau_1^*\}\mathbb{I}\{\varepsilon W_1^i\lambda_i \in [d_i^-, d_i^+]\}] \leq \frac{\beta^S}{\beta^S + m_\varepsilon\theta}\left(1 - \frac{m_\varepsilon}{\beta^S}\right), \\
&2. \quad \sum_{i=1}^{N+M} \mathbf{E}[e^{-\theta m_\varepsilon \tau_1^i}\mathbb{I}\{\tau_1^i = \tau_1^*\} \\
&\qquad \times \mathbb{I}\{\varepsilon\lambda_i W_1^i \notin [d_i^- + (1 + 2C)\varepsilon^\gamma, d_i^+ - (1 + 2C)\varepsilon^\gamma]\}] \\
&\quad \leq \frac{\beta^S}{\beta^S + m_\varepsilon\theta}\frac{m_\varepsilon}{\beta^S}(1 + C_5), \\
&3. \quad \sum_{i=1}^{N+M} \mathbf{E}[e^{-\theta m_\varepsilon \tau_1^i}\mathbb{I}\{\tau_1^i = \tau_1^*\}\mathbb{I}\{\varepsilon\lambda_i W_1^i \in [d_i^-, d_i^- + (2 + 2C)\varepsilon^\gamma]\} \\
&\qquad + \mathbb{I}\{\varepsilon\lambda_i W_1^i \in [d_i^+ - (2 + 2C)\varepsilon^\gamma, d_i^+]\}] \\
&\quad \leq \frac{\beta^S}{\beta^S + m_\varepsilon\theta}\frac{m_\varepsilon}{\beta^S}C_5.
\end{aligned} \tag{5.33}$$

6 Proof of Proposition 1: The Lower Bound

For $\theta > -1$ and $x \in \mathcal{G}_2$ we have the following estimate:

$$\mathbf{E}[e^{-\theta m_\varepsilon \sigma_\varepsilon}] \geq \sum_{k=1}^{\infty} \mathbf{E}[e^{-\theta m_\varepsilon \tau_k^*} \mathbb{I}\{\sigma_\varepsilon = \tau_k^*\}]. \tag{6.1}$$

For $k \in \mathbb{N}$ we further have

$$\begin{aligned}\mathbf{E}[e^{-\theta m_\varepsilon \tau_k^*} \mathbb{I}\{\sigma_\varepsilon = \tau_k^*\}] &= \mathbf{E}[e^{-\theta m_\varepsilon \tau_k^*} \mathbb{I}\{X_t^\varepsilon \in \mathcal{G}_1, t \in [0, \tau_k^*), X_{\tau_k^*}^\varepsilon \notin \mathcal{G}_1\}] \\ &= \mathbf{E}[e^{-\theta m_\varepsilon \tau_k^*} \mathbb{I}\{X_t^\varepsilon \in \mathcal{G}_1, t \in [0, \tau_1^*)\} \cdots \\ &\quad \times \mathbb{I}\{X_t^\varepsilon \in \mathcal{G}_1, t \in [\tau_{k-1}^*, \tau_k^*)\} \mathbb{I}\{X_{\tau_k^*}^\varepsilon \notin \mathcal{G}_1\}].\end{aligned} \tag{6.2}$$

The strong Markov property allows to write

$$\begin{aligned}&\mathbf{E}[e^{-\theta m_\varepsilon \tau_k^*} \mathbb{I}\{\sigma_\varepsilon = \tau_k^*\}] \\ &\quad = \mathbf{E}\Bigg[e^{-\theta m_\varepsilon \tau_k^*} \prod_{i=1}^{k-1} \mathbb{I}\{X_t^\varepsilon(X_{\tau_{i-1}^*}^\varepsilon) \in \mathcal{G}_1, t \in [0, S_i^*]\} \\ &\qquad \times \mathbb{I}\{X_t^\varepsilon(X_{\tau_{k-1}^*}) \in \mathcal{G}_1, t \in [0, S_k^*), X_{\tau_k^*}^\varepsilon \notin \mathcal{G}_1\}\Bigg].\end{aligned} \tag{6.3}$$

The preceding expression can be estimated by applying law properties of the small jump component v^k:

$$\begin{aligned}&\mathbf{E}\Bigg[e^{-\theta m_\varepsilon \tau_k^*} \prod_{i=1}^{k-1} \mathbb{I}\{X_t^\varepsilon(X_{\tau_{i-1}^*}) \in \mathcal{G}_1, t \in [0, S_i^*]\} \\ &\quad \times \mathbb{I}\{X_t^\varepsilon(X_{\tau_{k-1}^*}) \in \mathcal{G}_1, t \in [0, S_k^*), X_{\tau_k^*}^\varepsilon \notin \mathcal{G}_1\}\Bigg] \\ &= \mathbf{E}\Bigg[\prod_{i=1}^{k-1} e^{-\theta m_\varepsilon S_i^*} \mathbb{I}\{A^i(X_{\tau_{i-1}^*}^\varepsilon)\} e^{-\theta m_\varepsilon S_k^*} \mathbb{I}\{B^k(X_{\tau_{k-1}^*})\}\Bigg] \\ &\geq \left(\mathbf{E}\Big[e^{-\theta m_\varepsilon S_1^*} \inf_{y \in \mathcal{G}_2} \mathbb{I}\{A_1^-(y)\}\Big]\right)^{k-1} \mathbf{E}\Big[e^{-\theta m_\varepsilon S_1^*} \inf_{y \in \mathcal{G}_2} \mathbb{I}\{B^1(y)\}\Big].\end{aligned} \tag{6.4}$$

The two terms in the last preceding expression will be further estimated by decomposing them into terms reflecting jumps in the various directions. We estimate for $y \in \mathcal{G}_2$:

$$\mathbb{I}\{A_1^-(y)\} \geq \sum_{i=1}^{N+M} \mathbb{I}\{A_{1,i}^-(y)\} \mathbb{I}\{\tau_1^* = \tau_1^i\}. \tag{6.5}$$

Let $I_i = (d_i^- + (2+2C)\varepsilon^\gamma, d_i^+ - (2+2C)\varepsilon^\gamma)$. Using Lemmas 5 and 6 we obtain the following chain of inequalities for $A_{1,i}^-$, $i \in \{1, \dots, M+N\}$:

$$\begin{aligned}\mathbb{I}\{A_{1,i}^-(y)\} &\geq \mathbb{I}\{A_{1,i}^-(y)\}\mathbb{I}\{E\} \\ &\geq \mathbb{I}\{\varepsilon W_1^i \lambda_i \in I_i\} - 2 \cdot \mathbb{I}\{E^c\} \\ &\quad - \mathbb{I}\{\tau_1^i \leq T_R\}\mathbb{I}\{|\varepsilon\lambda_i W_1^i| \geq \varepsilon^\gamma\} - \mathbb{I}\{\tau_1^i < 5C^2\varepsilon^\gamma\}, \end{aligned} \tag{6.6}$$

for ε small enough such that $[-\varepsilon^\gamma, \varepsilon^\gamma] \subseteq I_i$.

Substituting this into (6.5) gives

$$\begin{aligned}\mathbb{I}\{A_1^-(y)\} \geq \sum_{i=1}^{N+M} \mathbb{I}\{\tau_1^* = \tau_1^i\}(\mathbb{I}\{\varepsilon W_1^i \lambda_i \in I_i\} - \mathbb{I}\{\tau_1^i < 5C^2\varepsilon^\gamma\} \\ - 2 \cdot \mathbb{I}\{E^c\} - \mathbb{I}\{\tau_1^i \leq T_R\}\mathbb{I}\{|\varepsilon\lambda_i W_1^i| \geq \varepsilon^\gamma\}). \end{aligned} \tag{6.7}$$

Hence by Propositions 3 and 4 there exits $K > 0$ and $\varepsilon_1 > 0$, such that for all $\varepsilon < \varepsilon_1$:

$$\mathbf{E}\Big[e^{-\theta m_\varepsilon \tau_1^*} \inf_{y \in \mathcal{G}_2} \mathbb{I}\{A_1^-(y)\}\Big] \geq \frac{\beta^S}{\beta^S + m_\varepsilon\theta}\Big(1 - \frac{m_\varepsilon}{\beta^S}(1+K)\Big). \tag{6.8}$$

For the sets $B^1(y)$, $y \in \mathcal{G}_2$, we use similar arguments. We have

$$\mathbb{I}\{B^1(y)\} = \sum_{i=1}^{N+M} \mathbb{I}\{B^{1,i}(y)\}\mathbb{I}\{\tau_1^* = \tau_1^i\}. \tag{6.9}$$

Lemma 5 allows to write

$$\begin{aligned}\mathbb{I}\{B^{1,i}(y)\} &\geq \mathbb{I}\{B^{1,i}(y)\}\mathbb{I}\{E\}\mathbb{I}\{\tau_1^i > T_R\} \\ &\geq \mathbb{I}\{\varepsilon W_1^i \lambda_i \notin [d_i^-, d_i^+]\}\mathbb{I}\{\tau_1^i > T_R\} - \mathbb{I}\{E^c\},\end{aligned}$$

for $i \in \{1, \ldots, M+N\}$.

If we insert this to (6.9) we see:

$$\mathbb{I}\{B^1(y)\} \geq \sum_{i=1}^{N+M} \mathbb{I}\{\tau_1^* = \tau_1^i\}(\mathbb{I}\{\varepsilon W_1^i \lambda_i \notin [d_i^-, d_i^+]\}\mathbb{I}\{\tau_1^i > T_R\} - \mathbb{I}\{E^c\}). \tag{6.10}$$

Again with (6.10) and Proposition 4 for $K > 0$ there exists $\varepsilon_2 > 0$, such that for all $\varepsilon \in (0, \varepsilon_2)$:

$$\mathbf{E}\Big[e^{-\theta m_\varepsilon \tau_1^*} \inf_{y \in \mathcal{G}_2} \mathbb{I}\{B^1(y)\}\Big] \geq \frac{\beta^S}{\beta^S + m_\varepsilon\theta}\frac{m_\varepsilon}{\beta^S}(1-K). \tag{6.11}$$

Finally for $K > 0$ we conclude with the help of (6.4), (6.8) and (6.11) that there exists an $\varepsilon_0 > 0$ with $0 < \varepsilon < \varepsilon_0 \leq \min(\varepsilon_1, \varepsilon_2)$ and small enough, that $\frac{m_\varepsilon}{\beta^S}(1+K) < 1$ holds, so that we have

$$\begin{aligned}\mathbf{E}[e^{-\theta m_\varepsilon \sigma_\varepsilon}] &\geq \sum_{k=1}^{\infty}\left[\frac{\beta^S}{\beta^S + m_\varepsilon\theta}\right]^k \left[1 - \frac{m_\varepsilon}{\beta^S}(1+K)\right]^{k-1}\left[\frac{m_\varepsilon}{\beta^S}(1-K)\right] \\ &= \frac{1-K}{\theta + 1 + K}.\end{aligned} \tag{6.12}$$

This justifies the claim. □

7 Proof of Proposition 2: The Upper Bound

As in the proof of Proposition 1 we decompose the Laplace transform into

$$\mathbf{E}[e^{-\theta m_\varepsilon \sigma_\varepsilon}] = \sum_{k=1}^{\infty} \underbrace{\mathbf{E}[[e^{-\theta m_\varepsilon \sigma_\varepsilon} \mathbb{I}\{\sigma_\varepsilon = \tau_k^*\}]}_{(*)} + Rest_k], \tag{7.1}$$

where

$$Rest_k \leq \begin{cases} \mathbf{E}[e^{-\theta m_\varepsilon \tau_k^*} \mathbb{I}\{\sigma_\varepsilon \in (\tau_{k-1}^*, \tau_k^*)\}], & \theta \in (-1, 0) \\ \mathbf{E}[e^{-\theta m_\varepsilon \tau_{k-1}^*} \mathbb{I}\{\sigma_\varepsilon \in (\tau_{k-1}^*, \tau_k^*)\}], & \theta \in [0, \infty). \end{cases} \tag{7.2}$$

Let us decompose $(*)$ similarly to the proof of Proposition 1. This yields the following estimates:

$$\begin{aligned}
&\mathbf{E}[e^{-\theta m_\varepsilon \sigma_\varepsilon} \mathbb{I}\{\sigma_\varepsilon = \tau_k^*\}] \\
&= \mathbf{E}[e^{-\theta m_\varepsilon \tau_k^*} \mathbb{I}\{X_t^\varepsilon \in \mathcal{G}_1, t \in [0, \tau_k^*); X_{\tau_k^*}^\varepsilon \notin \mathcal{G}_1\}] \\
&= \mathbf{E}\Bigg[e^{-\theta m_\varepsilon \tau_k^*} \prod_{i=1}^{k-1} \mathbb{I}\{X_t^\varepsilon(X_{\tau_{i-1}^*}^\varepsilon) \in \mathcal{G}_1, t \in [0, S_i^*)\} \\
&\quad \times \mathbb{I}\{X_t^\varepsilon(X_{\tau_{k-1}^*}^\varepsilon) \in \mathcal{G}_1; X_{\tau_k^*}^\varepsilon \notin \mathcal{G}_1\} \Bigg] \\
&= \mathbf{E}\Bigg[\prod_{i=1}^{k-1} e^{-\theta m_\varepsilon S_i^*} \mathbb{I}\{A^i(X_{\tau_{i-1}^*})\} e^{-\theta m_\varepsilon S_k^*} \mathbb{I}\{B^k(X_{\tau_{k-1}^*})\} \Bigg] \\
&\leq \underbrace{\Big[\mathbf{E}\Big[e^{-\theta m_\varepsilon \tau_1^*} \sup_{y \in \mathcal{G}_1} \mathbb{I}\{A^1(y)\}\Big]\Big]^{k-1}}_{(O1)} \underbrace{\mathbf{E}\Big[e^{-\theta m_\varepsilon \tau_1^*} \sup_{y \in \mathcal{G}_1} \mathbb{I}\{B^1(y)\}\Big]}_{(O2)}.
\end{aligned} \tag{7.3}$$

For $k = 1$ and $x \in \mathcal{G}_2$ we get in the cases $\theta \in (-1, 0)$ resp. $\theta > 0$

$$\begin{aligned}
Rest_1 &= \begin{cases} \mathbf{E}[e^{-\theta m_\varepsilon \tau_1^*} \mathbb{I}\{\sigma_\varepsilon \in (0, \tau_1^*)\}], \\ \mathbf{E}[\mathbb{I}\{\sigma_\varepsilon \in (0, \tau_1^*)\}], \end{cases} \\
&\leq \begin{cases} \mathbf{E}[e^{-\theta m_\varepsilon \tau_1^*} \mathbb{I}\{\exists t \in (0, S_1^*) : X_t^\varepsilon \notin \mathcal{G}_1\}], \\ \mathbf{E}[\mathbb{I}\{\exists t \in (0, S_1^*) : X_t^\varepsilon \notin \mathcal{G}_1\}], \end{cases} \\
&\leq \begin{cases} \mathbf{E}[e^{-\theta m_\varepsilon \tau_1^*} \sup_{y \in \mathcal{G}_2} \mathbb{I}\{\exists t \in (0, S_1^*) : v_t^1(y) \notin \mathcal{G}_1\}], \\ \mathbf{E}[\sup_{y \in \mathcal{G}_2} \mathbb{I}\{\exists t \in (0, S_1^*) : v_t^1(y) \notin \mathcal{G}_1\}]. \end{cases}
\end{aligned} \tag{7.4}$$

For $k \geq 2$ we obtain, in the cases $\theta \in (-1, 0)$ resp. $\theta > 0$

$$Rest_k = \begin{cases} \mathbf{E}[e^{-\theta m_\varepsilon \tau_k^*} \mathbb{I}\{\sigma_\varepsilon \in (\tau_{k-1}^*, \tau_k^*)\}], \\ \mathbf{E}[e^{-\theta m_\varepsilon \tau_{k-1}^*} \mathbb{I}\{\sigma_\varepsilon \in (\tau_{k-1}^*, \tau_k^*)\}], \end{cases}$$

$$
= \begin{cases} \mathbf{E}[e^{-\theta m_\varepsilon \tau_k^*}\mathbb{I}\{X_t^\varepsilon \in \mathcal{G}_1, t \in [0, \tau_{k-1}^*)\}\mathbb{I}\{\exists t \in (\tau_{k-1}^*, \tau_k^*) : X_t^\varepsilon \notin \mathcal{G}_1\}] \\ \mathbf{E}[e^{-\theta m_\varepsilon \tau_{k-1}^*}\mathbb{I}\{X_t^\varepsilon \in \mathcal{G}_1, t \in [0, \tau_{k-1}^*)\}\mathbb{I}\{\exists t \in (\tau_{k-1}^*, \tau_k^*) : X_t^\varepsilon \notin \mathcal{G}_1\}] \end{cases}
$$

$$
\le \begin{cases} [\mathbf{E}[e^{-\theta m_\varepsilon S_1^*} \sup_{y\in\mathcal{G}_1} \mathbb{I}\{A^1(y)\}]]^{k-2} \\ \mathbf{E}[e^{-\theta m_\varepsilon S_1^*} \sup_{y\in\mathcal{G}_1} \mathbb{I}\{A^1(y)\}\mathbb{I}\{\exists t \in (0, S_2^*) : v_t^2(X_{\tau_1^*}^\varepsilon + \varepsilon W_1^*) \notin \mathcal{G}_1\}] \\ [\mathbf{E}[e^{-\theta m_\varepsilon S_1^*} \sup_{y\in\mathcal{G}_1} \mathbb{I}\{A^1(y)\}]]^{k-2} \\ \mathbf{E}[e^{-\theta m_\varepsilon S_1^*} \sup_{y\in\mathcal{G}_1} \mathbb{I}\{A^1(y)\}\mathbb{I}\{\exists t \in (0, S_2^*) : v_t^2(X_{\tau_1^*}^\varepsilon + \varepsilon W_1^*) \notin \mathcal{G}_1\}\}. \end{cases} \tag{7.5}
$$

Estimation of the part (O1)

For $y \in \mathcal{G}_1$ we decompose

$$
\mathbb{I}\{A^1(y)\} = \sum_{i=1}^{N+M} \mathbb{I}\{A^{1,i}(y)\}\mathbb{I}\{\tau_1^* = \tau_1^i\}, \tag{7.6}
$$

which can be further estimated with Lemma 7 for $1 \le i \le M + N$ by

$$
\begin{aligned}
\mathbb{I}\{A^{1,i}(y)\} &\le \mathbb{I}\{A^{1,i}(y)\}\mathbb{I}\{E\} + \mathbb{I}\{E^c\} \\
&\le \mathbb{I}\left\{\varepsilon\lambda_i W_1^i \in [d_i^-, d_i^+]\right\}\mathbb{I}\{E\}\mathbb{I}\{|\varepsilon\lambda_i W_1^i| > \varepsilon^\gamma\}\mathbb{I}\{\tau_1^i \ge T_R\} \\
&\quad + \mathbb{I}\{|\varepsilon\lambda_i W_1^i| > \varepsilon^\gamma\}\mathbb{I}\{\tau_1^i < T_R\} + \mathbb{I}\{E^c\} + \mathbb{I}\{|\varepsilon\lambda_i W_1^i| \le \varepsilon^\gamma\}.
\end{aligned} \tag{7.7}
$$

If we choose ε small enough such that from $|\varepsilon\lambda_i W_1^i| \le \varepsilon^\gamma$ we can deduce $\varepsilon\lambda_i W_1^i \in [d_i^-, d_i^+]$ we get

$$
\begin{aligned}
\mathbb{I}\{A^{1,i}(y)\} &\le \mathbb{I}\left\{\varepsilon\lambda_i W_1^i \in [d_i^-, d_i^+]\right\} + \mathbb{I}\{E^c\} \\
&\quad + \mathbb{I}\{|\varepsilon\lambda_i W_1^i| > \varepsilon^\gamma\}\mathbb{I}\{\tau_1^i < T_R\}.
\end{aligned} \tag{7.8}
$$

With the help of Proposition 3 and 5 for every $K > 0$ there is an $\varepsilon_1 > 0$, such that for all $\varepsilon \in (0, \varepsilon_1)$ the inequality

$$
\mathbf{E}\left[e^{-\theta m_\varepsilon \tau_1^*} \sup_{y\in\mathcal{G}_1} \mathbb{I}\{A^1(y)\}\right] \le \frac{\beta^S}{\theta m_\varepsilon + \beta^S}\left[1 - \frac{m_\varepsilon}{\beta^S}(1 - K)\right]. \tag{7.9}
$$

holds.

Estimation of the part (O2)

Again for $y \in \mathcal{G}_1$ we have the decomposition

$$
\mathbb{I}\{B^1(y)\} = \sum_{i=1}^{N+M} \mathbb{I}\{B^{1,i}(y)\}\mathbb{I}\{\tau_1^i = \tau_1^*\}. \tag{7.10}
$$

With the help of Lemma 7 we see

$$
\begin{aligned}
\mathbb{I}\{B^{1,i}(y)\} &\le \mathbb{I}\{B^{1,i}(y)\}\mathbb{I}\{E\} + \mathbb{I}\{E^c\} \\
&\le \mathbb{I}\left\{\varepsilon\lambda_i W_1^i \notin [d_i^- + (1+2C)\varepsilon^\gamma, d_i^+ - (1+2C)\varepsilon^\gamma]\right\} \\
&\quad + \mathbb{I}\{|\varepsilon\lambda_i W_1^i| > \varepsilon^\gamma\}\mathbb{I}\{\tau_1^i < T_R\} + \mathbb{I}\{\mathcal{E}_T(\varepsilon,\gamma)^c\} + \mathbb{I}\{\tau_1^i \ge T_R\} \\
&\quad + \mathbb{I}\{\tau_1^i < 5C^2\varepsilon^\gamma\},
\end{aligned}
$$

where in the last step we use that $\mathbb{I}\{B^{1,i}(y)\}\mathbb{I}\{|\varepsilon\lambda_i W_1^i| \le \varepsilon^\gamma\}\mathbb{I}\{\tau_1^i \ge 5C^2\varepsilon^\gamma\} = 0$. For ε small enough we deduce for a constant $K > 0$

$$\mathbf{E}\Big[e^{-\theta m_\varepsilon \tau_1^*} \sup_{y\in\mathcal{G}_1} \mathbb{I}\{B^1(y)\}\Big] \le \frac{\beta^S}{\theta m_\varepsilon + \beta^S}\frac{m_\varepsilon}{\beta^S}\Big[1 + \frac{K}{3}\Big]. \tag{7.11}$$

Estimation of the remainder terms

To estimate $\mathbb{I}\{\exists t \in (0, S_1^*) : v_t^1(y) \notin \mathcal{G}_1\}$ for $y \in \mathcal{G}_2$, write

$$\mathbb{I}\{\exists t \in (0, S_1^*) : v_t^1(y) \notin \mathcal{G}_1\} \tag{7.12}$$

$$= \sum_{i=1}^{N+M} \mathbb{I}\{\exists t \in (0, S_1^*) : v_t^1(y) \notin \mathcal{G}_1\}\mathbb{I}\{\tau_1^* = \tau_1^i\} \tag{7.13}$$

$$\le \sum_{i=1}^{N+M} \mathbb{I}\{\tau_1^* = \tau_1^i\}\big[\mathbb{I}\{\exists t \in (0, S_1^*) : v_t^1(y) \notin \mathcal{G}_1\}\mathbb{I}\{E\} + \mathbb{I}\{E^c\}\big] \tag{7.14}$$

$$\le \mathbb{I}\{E^c\}. \tag{7.15}$$

For a $K > 0$ there exits $\varepsilon_3 > 0$, such that for all $0 < \varepsilon < \varepsilon_3$

$$\begin{aligned}\mathbf{E}\big[e^{-\theta m_\varepsilon \tau_1^*}\mathbb{I}\{\exists t \in (0, S_1^*) : v_t^1(y) \notin \mathcal{G}_1\}\big] &\le \mathbf{E}\big[e^{-\theta m_\varepsilon \tau_1^*}\mathbb{I}\{\mathcal{E}_T(\varepsilon,\gamma)^c\}\big]\\ &\le \frac{\beta^S}{\theta m_\varepsilon + \beta^S}\exp(-\varepsilon^{-p})\\ &\le \frac{\beta^S}{\theta m_\varepsilon + \beta^S}\frac{m_\varepsilon}{\beta^S}\Big(\frac{K}{3}\Big).\end{aligned} \tag{7.16}$$

For $k = 1$ we can conclude

$$Rest_1 \le \begin{cases} \frac{\beta^S}{\theta m_\varepsilon+\beta^S}\frac{m_\varepsilon}{\beta^S}(\frac{K}{3}), & \theta \in (-1, 0)\\ \exp(-\varepsilon^{-p}), & \theta \in [0, \infty]\end{cases} \le \frac{\beta^S}{\theta m_\varepsilon + \beta^S}\frac{m_\varepsilon}{\beta^S}\Big(\frac{K}{3}\Big). \tag{7.17}$$

In case $k \ge 2$ we may estimate for $y \in \mathcal{G}_1$:

$$\begin{aligned}&\mathbb{I}\{A^1(y)\}\mathbb{I}\{\exists t \in (0, S_2^*) : v_t^2(X^\varepsilon_{\tau_1^*} + \varepsilon W_1^*) \notin \mathcal{G}_1\}\\ &\quad\le \sup_{y\in\mathcal{G}_2}\mathbb{I}\{\exists t \in (0, S_2^*) : v_t^2(y) \notin \mathcal{G}_1\}\\ &\qquad + \mathbb{I}\big\{v_t^1(y) \in \mathcal{G}_1, t \in [0, \tau_1^*], v^1_{\tau_1^*} + \varepsilon W_1^* \in \mathcal{G}_1\backslash\mathcal{G}_2\big\}\\ &\quad\le \sup_{y\in\mathcal{G}_2}\mathbb{I}\{\exists t \in (0, S_1^*) : v_t^1(y) \notin \mathcal{G}_1\}\\ &\qquad + \mathbb{I}\big\{v_t^1(y) \in \mathcal{G}_1, t \in [0, \tau_1^*], v^1_{\tau_1^*} + \varepsilon W_1^* \in \mathcal{G}_1\backslash\mathcal{G}_2\big\}.\end{aligned} \tag{7.18}$$

The first term is identical to the expression for $k = 1$. The second part is treated as (O1), where $\mathbb{I}\{A^1(y)\}$ is replaced with $\mathbb{I}\{\tilde{A}^1(y)\}$.

Therefore, in case $\theta \geq 0$ for $K > 0$ there exists an $\varepsilon_4 > 0$ such that for all $0 < \varepsilon < \varepsilon_4$:

$$\begin{aligned}
&\mathbf{E}\Big[e^{-\theta\gamma\tau_1^*} \sup_{y\in\mathcal{G}_1} \mathbb{I}\{A^1(y)\}\mathbb{I}\{\exists t \in (0, S_2^*) : v_t^2(X^\varepsilon_{\tau_1^*} + \varepsilon W_1^*) \notin \mathcal{G}_1\}\Big] \\
&\quad \leq \frac{\beta^S}{\theta m_\varepsilon + \beta^S}\left(2\exp(-\varepsilon^{-p}) + \frac{m_\varepsilon}{\beta^S} C_5 + C_1 \varepsilon^{\alpha_1(1-\rho-\gamma)}(\beta^S)|\ln\varepsilon|\right) \\
&\quad \leq \frac{\beta^S}{\theta m_\varepsilon + \beta^S}\frac{m_\varepsilon}{\beta^S}\frac{K}{3}.
\end{aligned} \tag{7.19}$$

Analogously, for $\theta \in (-1, 0)$ and $K > 0$ there is an $\varepsilon_5 > 0$, such that for all $0 < \varepsilon < \varepsilon_5$:

$$\begin{aligned}
&\mathbf{E}\big[e^{-\theta\gamma\tau_1^* + \tau_2^*} \sup_{y\in\mathcal{G}_1} \mathbb{I}\{A^1(y)\}\mathbb{I}\{\exists t \in (0, S_2^*) : v_t^2(X^\varepsilon_{\tau_1^*} + \varepsilon W_1^*) \notin \mathcal{G}_1\}\big] \\
&\quad \leq \left(\frac{\beta^S}{\theta m_\varepsilon + \beta^S}\right)^2 \frac{m_\varepsilon}{\beta^S}\frac{K}{4} \\
&\quad \leq \frac{\beta^S}{\theta m_\varepsilon + \beta^S}\frac{m_\varepsilon}{\beta^S}\frac{K}{3}.
\end{aligned} \tag{7.20}$$

If we choose $\varepsilon > 0$ so that $0 < \varepsilon < \varepsilon_0 \leq \min(\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5)$ and that $\frac{\beta^S}{\theta m_\varepsilon + \beta^S}[1 - \frac{m_\varepsilon}{\beta^S}(1 - C)] < 1$, we may estimate with (7.9), (7.11), (7.17), (7.19) and (7.20) for $K \in (0, \theta + 1)$ to obtain the desired inequality

$$\begin{aligned}
&\mathbf{E}\big[e^{-\theta m_\varepsilon \sigma_\varepsilon}\big] \\
&\quad \leq \sum_{k=1}^{\infty}\left(\frac{\beta^S}{\theta m_\varepsilon + \beta^S}\left[1 - \frac{m_\varepsilon}{\beta^S}(1-K)\right]\right)^{k-1} \frac{\beta^S}{\theta m_\varepsilon + \beta^S}\frac{m_\varepsilon}{\beta^S}\left(1 + \frac{K}{3}\right) \\
&\qquad + \frac{\beta^S}{\theta m_\varepsilon + \beta^S}\frac{m_\varepsilon}{\beta^S}\frac{C}{3} + \sum_{k=2}^{\infty}\left(\frac{\beta^S}{\theta m_\varepsilon + \beta^S}\left[1 - \frac{m_\varepsilon}{\beta^S}(1-K)\right]\right)^{k-2} \frac{\beta^S}{\theta m_\varepsilon + \beta^S}\frac{m_\varepsilon}{\beta^S}\frac{K}{3} \\
&\quad = \frac{1+K}{\theta + 1 - K}.
\end{aligned} \tag{7.21}$$

This completes the proof. □

To obtain Theorem 1 we conclude directly from Propositions 1 and 2 that for $K \in (0, \theta + 1)$:

$$\frac{1-K}{\theta+1+K} \leq \mathbf{E}e^{-\theta m_\varepsilon \sigma_\varepsilon} \leq \frac{1+K}{\theta+1-K}. \tag{7.22}$$

Since K can be chosen arbitrarily, we see

$$\lim_{\varepsilon\downarrow 0}\mathbf{E}e^{-\theta m_\varepsilon \sigma_\varepsilon} = \frac{1}{\theta+1}. \tag{7.23}$$

Convergence of moments and exit probabilities

Since the Laplace transform is finite for $-1 < \theta < 0$, the random variable $m_\varepsilon\sigma_\varepsilon$ has moments of all orders. Since the family $\{(m_\varepsilon\sigma_\varepsilon)^p\}_{0<\varepsilon<\varepsilon_0}$ of positive random variables is uniformly integrable, we obtain the convergence of the moments.

Let all the constants defined as in (3.7). Redefining the sets B^1 and $B^{1,i}$ by

$$C_+^1(y) := \{v_t^1(y) \in \mathcal{G}_1, t \in [0, S_1^*), v_{\tau_1^*}^k(y) + \varepsilon W_1^* \in \Omega_i^{\varepsilon+}(\delta)\}, \tag{7.24}$$

$$C_+^{j,i}(y) := \{v_t^1(y) \in \mathcal{G}_1, t \in [0, \tau_1^j), v_{\tau_1^j}^1(y) + \varepsilon\lambda_j r_j W_1^j \in \Omega_i^{\varepsilon+}(\delta)\} \tag{7.25}$$

and setting $\theta = 0$, we may repeat the arguments of the proofs of Propositions 1 and 2 taking (C3) into account become the asymptotics of the exit probabilities (4.6). □

8 Examples

One-dimensional case

On an interval $[-b, a]$, where $a, b > 0$ we consider the stochastic differential equation

$$X_t^\varepsilon(x) = x - \int_0^t U'(X_t^\varepsilon(x))ds + \varepsilon L_t \quad (\varepsilon > 0). \tag{8.1}$$

Here L is a Lévy process with an α-stable component and $U \in C^1$ is a 'parabolic'-shaped potential function, i.e. U has a global minimum at the origin, $U'(x)x \geq 0$, $U(0) = 0$, $U'(x) = 0$ iff $x = 0$ and $U''(0) = M > 0$. This guarantees the existence of a unique solution, that 0 is an exponentially stable point and an attractor of the domain. So all the required assumptions on the domain and the vector field are fulfilled. Thus, the exit time $\sigma(\varepsilon) = \inf\{t > 0 : X_t^\varepsilon \notin [-b, a]\}$ is approximately exponentially distributed with parameter $\frac{\varepsilon^\alpha}{\alpha}[\frac{1}{a^\alpha} + \frac{1}{b^\alpha}]$. For the probability on which side of the interval the process exits, we have

$$\begin{aligned} \lim_{\varepsilon\to 0} \mathbf{P}(X_{\sigma_\varepsilon}^\varepsilon > a) &= \frac{a^{-\alpha}}{a^{-\alpha} + b^{-\alpha}}, \\ \lim_{\varepsilon\to 0} \mathbf{P}(X_{\sigma_\varepsilon}^\varepsilon < -b) &= \frac{b^{-\alpha}}{a^{-\alpha} + b^{-\alpha}}. \end{aligned} \tag{8.2}$$

This result is exactly the one obtained in [13].

Ornstein-Uhlenbeck-type processes

We consider the ball of radius R in $\mathbb{R}^d$ and for $\theta > 0$ the SDE

$$dX_t^\varepsilon = -\theta X_t^\varepsilon + \varepsilon(r_1 dL_t^1 + r_2 dL_t^2) \quad (\varepsilon > 0), \tag{8.3}$$

where L^1 and L^2 are Lévy processes with α-stable components. Since all the assumptions are fulfilled, we know that the exit time is approximately exponential distributed with parameter $\frac{4\varepsilon^\alpha}{\alpha R^\alpha}$ and the probability of leaving the ball in each of the four directions is $\frac{1}{4}$.

Jumps dominate the Brownian motion

This example deals with the intuition that if the system is perturbed by a Lévy noise with stable component and a Brownian motion, in the small limit of ε the Lévy noise will dominate its exit behavior. For this purpose we define the dynamical system

$$dX_t^\varepsilon = b(X_t^\varepsilon) + \varepsilon(\lambda_1 r_1 dB_t + \lambda_2 r_2 dL_t), \tag{8.4}$$

where L is a Lévy process with α-stable component as before, B an independent standard Brownian motion and b a vector field that fulfills the conditions above. Let us define

$\tilde{\xi}_t := \varepsilon(\lambda_1 r_1 B_t + \lambda_2 r_2 \xi_t)$, where ξ is the small jump process of L. We can easily repeat the arguments of lemma 3 to show

$$\mathbf{P}\Big(\sup_{t\in[0,T_\varepsilon]} \|\varepsilon\tilde{\xi}_t\| \geq \varepsilon^{q\gamma}\Big) \leq \exp(-\varepsilon^{-p}). \tag{8.5}$$

This means that we can consider the Brownian motion to be part of the small jump part. Only the big jumps of L are important for the exit.

9 Conclusions

In this paper we study random perturbations of multidimensional deterministic dynamical systems with a stable attractor at the origin by white Lévy noise of small amplitude. The Lévy noise is assumed to be multifractal, i.e. it is composed of a finite sum of independent symmetric one-dimensional α_i-stable processes with laws being supported on straight lines spanned on unit vectors r_i. In the small amplitude limit we solve the problem of the first exit of the perturbed system from a bounded domain around the origin. We prove that the first exit time is exponentially distributed, determine the explicit formula for its mean value (Kramers' time) and in general all higher moments. Furthermore we determine the asymptotic law of the location of the first exit. It turns out that the exit is essentially controlled by the Lévy noise with the smallest stability index $\alpha_{\min}$ and the spatial geometry of the domain, in particular the length of the segments of the straight line spanned on the vector $r_{\alpha_{\min}}$ connecting the stable attractor and the domainn's border. Our proof is based on a simultaneous decomposition of the driving processes into appropriate small and big jump parts.

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